

Remarks on magnetic and electric Aharonov-Bohm effects

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Abstract

We give a direct proof of the magnetic Aharonov-Bohm effects without using the scattering theory and the theory of inverse boundary value problems. This proof can serve as a framework for a physical experiment to confirm the magnetic AB effect. We prove also the electric AB effect and we suggest a physical experiment to demonstrate the electric AB effect. In addition, we consider a combined electric and magnetic AB effect and we propose a new inverse problem for the time-dependent Schrödinger equations. Finally we study the gravitational AB effect.

1 Introduction.

Let Ω_1 be a bounded domain in \mathbf{R}^2 , called the obstacle. Consider the time-dependent Schrödinger equation in $(\mathbf{R}^2 \setminus \overline{\Omega}_1) \times (0, T)$:

$$(1.1) \quad -ih \frac{\partial u}{\partial t} + \frac{1}{2m} \sum_{j=1}^n \left(-ih \frac{\partial}{\partial x_j} - \frac{e}{c} A_j(x) \right)^2 u + eV(x)u = 0,$$

where $n = 2$,

$$(1.2) \quad u|_{\partial\Omega_1 \times (0, T)} = 0,$$

$$(1.3) \quad u(x, 0) = u_0(x), \quad x \in \mathbf{R}^2 \setminus \Omega_1,$$

and the electromagnetic potentials $A(x), V(x)$ are independent of t . Let

$$(1.4) \quad \alpha = \frac{e}{hc} \int_{\gamma} A(x) \cdot dx$$

be the magnetic flux, where γ is a simple closed contour containing Ω_1 . In seminal paper [AB] Y. Aharonov and D. Bohm discovered that even if the magnetic field $B(x) = \text{curl } A = 0$ in $\mathbf{R}^2 \setminus \overline{\Omega}_1$, the magnetic potential A has a physical impact in $\mathbf{R}^2 \setminus \overline{\Omega}_1$ when $\alpha \neq 2\pi n$, n is an integer. This phenomenon is called the Aharonov-Bohm effect. They proposed a physical experiment to test this effect. The experimental proof of AB effect was not easy to achieve. The most “clean” AB experiment was done by Tonomura et al [T et al]. A rigorous mathematical justification of the Tonomura et al experiment was given in [BW2], [BW3].

In the same paper [AB] Aharonov and Bohm gave a mathematical proof of AB effect by showing that the scattering cross section depends on α . The proofs of AB effects using the scattering theory and the inverse scattering were given also in [R], [N], [W1], [BW1], [RY1], [RY2], [Y], [EI], [EI0]. The inverse boundary value problems approach to AB effect was developed in [E1], [E2], [E3], [E4], [E5]. Note that the solution of the inverse scattering problem can be reduced to the solution of the inverse boundary value problem. This reduction is well-known in the case $n \geq 3$ and electromagnetic fields with compact supports (see, for example, [E5], §1). In the case $n = 2$ the reduction was proven in [EIO]. Note that the class of electromagnetic fields with compact supports is the natural setting for the study of AB effect.

In this paper we give a new direct mathematical proof of the magnetic AB effect that uses the relation between the solutions of the Schrödinger equation and the wave equation (see [K]). This proof can be used as a framework for a physical experiment to verify the magnetic AB effect. We consider the case of one and several obstacles. The case of several obstacles requires a technique of broken rays. It allows to detect the magnetic AB effect in the case when obstacles are close to each other and the treatment of the cluster of obstacles as one obstacle may miss the AB effect. We give also a rigorous proof of the electric AB effect and propose a physical experiment to verify it. We show that the electric AB effect occurs only when the domain is time-dependent with its topology changing in time. In addition we consider combined electric and magnetic AB effect, and gravitational AB effect

The plan of the paper is the following:

In §2 we state the AB effect. In §3 we prove the magnetic AB effect in the case of one obstacle in two and three dimensions and in the case of several obstacles. In §4 we consider the electric AB effect, and in §5 the combined electric and magnetic AB effect for time-dependent electromagnetic potentials. In the end of §5 we study a new inverse problem for the time-dependent Schrödinger equations. In §6 we prove a general case of the gravitational AB effect. A particular case was considered previously in [S].

2 The magnetic AB effect

Let $\Omega_1, \dots, \Omega_m$ be smooth obstacles in \mathbf{R}^n . Assume that $\overline{\Omega}_j \cap \overline{\Omega}_k \neq \emptyset$ if $j \neq k$. Consider the Schrödinger equation (1.1) in $(\mathbf{R}^n \setminus \overline{\Omega}) \times (0, T)$, $n \geq 2$, where $\Omega = \cup_{j=1}^m \Omega_j$,

$$(2.1) \quad u|_{\partial\Omega \times (0, T)} = 0,$$

and (1.3) holds in $\mathbf{R}^n \setminus \Omega$.

In (1.1) $A(x) = (A_1(x), \dots, A_n(x))$ is the magnetic potential and $V(x)$ is the electric potential.

In this paper we assume that $B(x) = \text{curl } A(x) = 0$ in $\mathbf{R}^n \setminus \overline{\Omega}$, i.e. the magnetic field $B(x)$ is shielded inside Ω . For the simplicity we assume that $V(x)$ has a compact support.

Denote by $G(\mathbf{R}^n \setminus \Omega)$ the group of C^∞ complex-valued functions $g(x)$ such that $|g(x)| = 1$ in $\mathbf{R}^n \setminus \Omega$ and

$$\begin{aligned} g(x) &= 1 + O\left(\frac{1}{|x|}\right) \quad \text{for } |x| > R \quad \text{if } n \geq 3, \\ g(x) &= e^{ip\theta(x)} \left(1 + O\left(\frac{1}{|x|}\right)\right) \quad \text{for } |x| > R \quad \text{if } n = 2. \end{aligned}$$

Here p is an arbitrary integer, $0 \in \Omega$ and $\theta(x)$ is the polar angle of x . We call $G(\mathbf{R}^n \setminus \Omega)$ the gauge group. If $u'(x) = g^{-1}(x)u(x)$ then $u'(x)$ satisfies the Schrödinger equation (1.1) with electromagnetic potentials $(A'(x), V'(x))$ where

$$(2.2) \quad \begin{aligned} V'(x) &= V(x), \\ \frac{e}{c}A'(x) &= \frac{e}{c}A(x) + ihg^{-1}(x)\frac{\partial g(x)}{\partial x}. \end{aligned}$$

We shall call electromagnetic potentials (A', V') and (A, V) gauge equivalent if there exists $g(x) \in G(\mathbf{R}^n \setminus \Omega)$ such that (2.2) holds.

We shall describe all gauge equivalence classes of potentials when $B = \text{curl } A = 0$ in $\mathbf{R}^n \setminus \Omega$.

Consider first the case of the obstacle Ω_1 in \mathbf{R}^2 . The gauge group $G(\mathbf{R}^2 \setminus \Omega_1)$ consists of $g(x) = e^{ip\theta(x) + \frac{i}{h}\varphi(x)}$, where p is an integer, and $\varphi(x) \in C^\infty(\mathbf{R}^2 \setminus \Omega_1)$, $\varphi(x) = O(\frac{1}{|x|})$ when $|x| > R$. The gauge equivalence class is determined by the magnetic flux (1.4) modulo $2\pi p$, $p \in \mathbf{Z}$.

In the case of several obstacles $\Omega_1, \dots, \Omega_m$ in \mathbf{R}^2 denote by γ_j , $1 \leq j \leq m$, a simple closed curve encircling Ω_j only. Let

$$\alpha_j = \frac{e}{hc} \int_{\gamma_j} A \cdot dx$$

be the corresponding magnetic flux. Then numbers $\alpha_j \pmod{2\pi n}$, $j = 1, \dots, m$, determine the gauge equivalent class of (A, V) .

Finally, in the case of $m \geq 1$ in \mathbf{R}^n , $n \geq 3$, it can be shown that there exists a finite number of closed curves $\gamma_1, \dots, \gamma_r$ in $\mathbf{R}^n \setminus \overline{\Omega}$ ($r = 0$ if $\mathbf{R}^n \setminus \Omega$ is simply-connected) such that (A, V) and (A', V') are gauge equivalent iff $\frac{e}{hc} \int_{\gamma_j} A \cdot dx - \frac{e}{hc} \int_{\gamma_j} A' \cdot dx = 2\pi n_j$, $n_j \in \mathbf{Z}$, for all $1 \leq j \leq r$. \square

Any two electromagnetic potentials belonging to the same gauge equivalence class represent the same physical reality and can not be distinguished in any physical experiment.

The Aharonov-Bohm effect is the statement that electromagnetic potentials belonging to different gauge equivalence classes have a different physical impact. Consider, for example, the probability density $|u(x)|^2$. It has the same value for any representative of the same gauge equivalence class since $|g^{-1}(x)u(x)|^2 = |u(x)|^2$.

To prove the AB effect it is enough to show that $|u(x)|^2$ changes for some $u(x)$ when we change the gauge equivalence class.

3 The proof of the magnetic AB effect

3.1 The case of one obstacle in \mathbf{R}^2

Consider the Schrödinger equation (1.1) in $(\mathbf{R}^2 \setminus \Omega_1) \times (0, T)$ with the boundary condition (1.2) and the initial condition (1.3).

Let $w(x, t)$ be the solution of the wave equation

$$(3.1) \quad \frac{h^2}{2m} \frac{\partial^2 w}{\partial t^2} + Hw = 0 \quad \text{in } (\mathbf{R}^2 \setminus \Omega_1) \times (0, +\infty)$$

with the boundary condition

$$(3.2) \quad w|_{\partial\Omega_1 \times (0, +\infty)} = 0$$

and the initial conditions

$$(3.3) \quad w(x, 0) = u_0(x), \quad \frac{\partial w(x, 0)}{\partial t} = 0, \quad x \in \mathbf{R}^2 \setminus \Omega_1,$$

i.e. $w(x, t)$ is even in t . Here

$$H = \frac{1}{2m} \left(-ih \frac{\partial}{\partial x} - \frac{e}{c} A \right)^2 + eV(x).$$

There is a formula relating $u(x, t)$ and $w(x, t)$ (cf. [K]):

$$(3.4) \quad u(x, t) = \frac{e^{-i\frac{\pi}{4}} \sqrt{m}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t}} w(x, x_0) dx_0.$$

We shall consider solutions of (3.1) such that

$$(3.5) \quad |w(x, t)| \leq C(1 + |t|)^m, \quad \left| \frac{\partial^r w(x, t)}{\partial t^r} \right| \leq C_r(1 + |t|)^m, \quad \forall r \geq 1.$$

Let $\chi_0(t) \in C_0^\infty(\mathbf{R}^1)$, $\chi_0(-t) = \chi_0(t)$, $\chi_0(t) = 1$ for $|t| < \frac{1}{2}$, $\chi_0(t) = 0$ for $|t| > 1$. We define the integral (3.4) as the limit of

$$(3.6) \quad \frac{e^{-i\frac{\pi}{4}} \sqrt{m}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} \chi_0(\varepsilon x_0) e^{\frac{imx_0^2}{2\hbar t}} w(x, x_0) dx_0$$

as $\varepsilon \rightarrow 0$, and we shall show that this limit exists for any $w(x, x_0)$ satisfying (3.5). Substitute the identity

$$\left(\frac{\hbar t}{imx_0} \frac{\partial}{\partial x_0} \right)^M e^{\frac{imx_0^2}{2\hbar t}} = e^{\frac{imx_0^2}{2\hbar t}}, \quad \forall M,$$

in (3.6) and integrate by parts in (3.6) for $|x_0| > 1$. If $M \geq m + 2$ we get an absolutely integrable function of x_0 and therefore we can pass to the limit when $\varepsilon \rightarrow 0$.

Note that

$$\left(-ih\frac{\partial}{\partial t} + H\right)u(x, t) = \frac{e^{-i\frac{\pi}{4}\sqrt{m}}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t}} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_0^2} + H\right)w(x, x_0)dx_0,$$

Note also that

$$u(x, 0) = \lim_{t \rightarrow 0} \frac{e^{-i\frac{\pi}{4}\sqrt{m}}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t}} w(x, x_0)dx_0 = w(x, 0).$$

Therefore $u(x, t)$ satisfies (1.1), (1.2), (1.3) if $w(x, t)$ satisfies (3.1), (3.2), (3.3). \square

We shall construct geometric optics type solutions of (3.1), (3.2), (3.3) and then use the formula (3.4) to obtain solutions of the Schrödinger equation.

We shall look for $w(x, t)$ in the form

$$(3.7) \quad w_N(x, t) = e^{i\frac{mk}{\hbar}(x \cdot \omega - t)} \sum_{p=0}^N \frac{a_p(x, t)}{(ik)^p} + e^{i\frac{mk}{\hbar}(x \cdot \omega + t)} \sum_{p=0}^N \frac{b_p(x, t)}{(ik)^p},$$

where k is a large parameter.

Substituting (3.7) into (3.1) and equating equal powers of k we get

$$(3.8) \quad \begin{aligned} \hbar a_{0t}(x, t) + \hbar \omega \cdot a_{0x}(x, t) - i\omega \cdot \frac{e}{c} A(x) a_0 &= 0, \\ -\hbar b_{0t}(x, t) + \hbar \omega \cdot b_{0x} - i\omega \cdot \frac{e}{c} A(x) b_0 &= 0, \end{aligned}$$

$$(3.9) \quad \begin{aligned} \hbar a_{pt}(x, t) + \hbar \omega \cdot a_{px}(x, t) - i\omega \cdot \frac{e}{c} A(x) a_p &= i \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial t^2} + H \right) a_{p-1}, \\ -\hbar b_{pt}(x, t) + \hbar \omega \cdot b_{px} - i\omega \cdot \frac{e}{c} A(x) b_p &= i \left(\frac{\partial^2}{\partial t^2} + H \right) b_{p-1}, \quad 1 \leq p \leq N. \end{aligned}$$

We have $b_p(x, t) = a_p(x, -t)$ for $p \geq 0$, assuming that $b_p(x, 0) = a_p(x, 0)$.

Introduce new coordinates (s, τ, t) instead of (x_1, x_2, t) where

$$(3.10) \quad \begin{aligned} s &= (x - x^{(0)}) \cdot \omega - t, \\ \tau &= (x - x^{(0)}) \cdot \omega_{\perp}, \\ t &= t. \end{aligned}$$

Here $\omega_\perp \cdot \omega = 0$, $|\omega_\perp| = |\omega| = 1$. We assume that $x^{(0)}$ is a fixed point outside of the obstacle Ω_1 and that the line $x = x^{(0)} + s\omega$, $s \in \mathbf{R}$, does not intersect Ω_1 . Equations (3.8), (3.9) have the following form in the new coordinates

$$(3.11) \quad \begin{aligned} \hat{a}_{0t}(s, \tau, t) - i\omega \cdot \frac{e}{\hbar c} A(x^{(0)} + (s+t)\omega + \tau\omega_\perp) \hat{a}_0 &= 0, \\ \hat{a}_{pt}(s, \tau, t) - i\omega \cdot \frac{e}{\hbar c} A(x^{(0)} + (s+t)\omega + \tau\omega_\perp) \hat{a}_p &= \hat{f}_p(s, \tau, t), \quad p \geq 1, \end{aligned}$$

where $\hat{a}_p(s, \tau, t) = a_p(x, t)$, $\hat{f}_p(s, \tau, t)$ is $\frac{i}{\hbar} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial t^2} + H \right) a_{p-1}$ in the new coordinates.

We impose the following initial conditions

$$(3.12) \quad \begin{aligned} \hat{a}_0(s, \tau, 0) &= \frac{1}{2} \chi_0\left(\frac{\tau}{\delta_1}\right) \chi_0\left(\frac{s}{\delta_2 k}\right), \\ \hat{a}_p(s, \tau, 0) &= 0 \quad \text{for } p \geq 1, \end{aligned}$$

where $\chi_0(s)$ is the same as above. We assume that δ_1 is such that $\text{supp } \chi_0\left(\frac{(x-x^{(0)}) \cdot \omega_\perp}{\delta_1}\right)$ does not intersect Ω_1 . Then

$$\hat{a}_0(s, \tau, t) = \frac{1}{2} \chi_0\left(\frac{\tau}{\delta_1}\right) \chi_0\left(\frac{s}{\delta_2 k}\right) \exp\left(\frac{ie}{\hbar c} \int_0^t \omega \cdot A(x^{(0)} + (s+t')\omega + \tau\omega_\perp) dt'\right).$$

Since $s = (x - x^{(0)}) \cdot \omega - t$ we have in the original coordinates

$$(3.13) \quad \begin{aligned} a_0(x, t) &= \frac{1}{2} \chi_0\left(\frac{(x - x^{(0)}) \cdot \omega_\perp}{\delta_1}\right) \chi_0\left(\frac{(x - x^{(0)}) \cdot \omega - t}{\delta_2 k}\right) \\ &\quad \cdot \exp\left(\frac{ie}{\hbar c} \int_0^t \omega \cdot A(x - t''\omega) dt''\right), \end{aligned}$$

where we made the change of variables $t - t' = t''$. Note that

$$(3.14) \quad |a_p(x, t)| \leq Ct^p, \quad 1 \leq p \leq N,$$

and (3.5) holds for any $r \geq 1$. Since $b_p(x, t) = a_p(x, -t)$, $p \geq 0$, we have that

$$(3.15) \quad \begin{aligned} w_N(x, 0) &= \chi_0\left(\frac{(x - x^{(0)}) \cdot \omega_\perp}{\delta_1}\right) \chi_0\left(\frac{(x - x^{(0)}) \cdot \omega}{\delta_2 k}\right) e^{i\frac{m}{\hbar} k\omega \cdot x}, \\ w_{Nt}(x, 0) &= 0. \end{aligned}$$

Let

$$(3.16) \quad u_N(x, t) = \frac{e^{-i\frac{\pi}{4}}\sqrt{m}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t}} w_N(x, x_0) dx_0.$$

Using that $b_p(x, t) = a_p(x, -t)$ and making a change of variables we get

$$(3.17) \quad u_N(x, t) = \frac{2e^{-i\frac{\pi}{4}}\sqrt{m}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t} + \frac{imk}{h}(x \cdot \omega - x_0)} \sum_{p=0}^N \frac{a_p(x, x_0)}{(ik)^p} dx_0.$$

We have

$$(3.18) \quad \left(-i\hbar \frac{\partial}{\partial t} + H\right) u_N(x, t) = \frac{e^{-i\frac{\pi}{4}}\sqrt{m}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t}} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_0^2} + H\right) w_N(x, x_0) dx_0.$$

Note that

$$\begin{aligned} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_0^2} + H\right) w_N(x, x_0) &= e^{\frac{imk}{h}(x \cdot \omega - x_0)} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_0^2} + H\right) a_N(x, x_0) \\ &\quad + e^{\frac{imk}{h}(x \cdot \omega + x_0)} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_0^2} + H\right) b_N(x, x_0). \end{aligned}$$

Denote by $g_N(x, t)$ the right hand side of (3.18). Since $b_N(x, x_0) = a_N(x, -x_0)$ we have

$$(3.19) \quad g_N(x, t) = \frac{2e^{-i\frac{\pi}{4}}\sqrt{m}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t}} e^{\frac{imk}{h}(x \cdot \omega - x_0)} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_0^2} + H\right) \frac{a_N(x, x_0)}{(ik)^N} dx_0.$$

We apply the stationary phase method to the integrals (3.17). The equation for the critical point is $\frac{mx_0}{\hbar t} - \frac{mk}{h} = 0$, i.e. $x_0 = kt$ and the Hessian is $\frac{m}{\hbar t}$. Therefore

$$(3.20) \quad u_N(x, t) = e^{-\frac{imk^2 t}{2\hbar} + i\frac{mk}{h}x \cdot \omega} \chi_0\left(\frac{(x - x^{(0)}) \cdot \omega_{\perp}}{\delta_1}\right) \exp\left(\frac{ie}{\hbar c} \int_0^{\infty} \omega \cdot A(x - s'\omega) ds'\right) + O(\varepsilon),$$

where ε is arbitrary small when k is sufficiently large and t is sufficiently small.

We used that $\chi_0\left(\frac{x \cdot \omega - kt}{\delta_2 k}\right) = 1$ when k is large and t is small. Note that

$$(3.21) \quad \left| \frac{a_p(x, kt)}{(ik)^p} \right| \leq \frac{C}{k^p} (kt)^p \leq Ct^p$$

is small when t is small.

Applying the stationary phase method to (3.19) we get, using (3.14) that

$$(3.22) \quad \int_{\mathbf{R}^2 \setminus \Omega} |g_N(x, t)|^2 dx \leq Ct^N k^{\frac{1}{2}}.$$

We used in (3.22) that $\chi_0\left(\frac{x \cdot \omega - kt}{\delta_2 k}\right) = 0$ when $|x \cdot \omega| > Ck$.

Denote by $\|g_N\|_r$ the Sobolev norm in $H_r(\mathbf{R}^2 \setminus \Omega_1)$. It follows from (3.19) and (3.14) that

$$\|g_N\|_r \leq Ct^N k^{r+\frac{1}{2}}.$$

Let $R_N(x, t)$ be the solution of

$$\left(-ih \frac{\partial}{\partial t} + H \right) R_N = -g_N(x, t) \quad \text{in } (\mathbf{R}^2 \setminus \Omega_1) \times (0, T),$$

$$R_N \Big|_{\partial \Omega_1 \times (0, T)} = 0,$$

$$R_N(x, 0) = 0.$$

Such solution exists and satisfies the following estimates (cf. [E1]):

$$(3.23) \quad \max_{0 \leq t \leq T} \|R_N(\cdot, t)\|_3 \leq C \int_0^T \left(\|g_N(\cdot, t)\|_1 + \left\| \frac{\partial g_N(\cdot, t)}{\partial t} \right\|_0 \right) dt.$$

By the Sobolev embedding theorem $|R_N(x, t)| \leq C \max_{0 \leq t \leq T} \|R_N(\cdot, t)\|_3$ for all $(x, t) \in (\mathbf{R}^2 \setminus \Omega_1) \times (0, T)$. Since

$$\max_{0 \leq t \leq T} \left\| \frac{\partial^p}{\partial t^p} g_N(x, t) \right\|_r \leq CT^N k^{\frac{1}{2}+r}$$

we get that

$$|R_N(x, t)| \leq C\varepsilon$$

if $T \leq \frac{C}{k^{\delta_3}}$, $0 < \delta_3 < 1$, $N\delta_3 > \frac{3}{2}$.

Note that $u = u_N + R_N$ satisfies (1.1), (1.2) and the initial condition $u(x, 0) = u_N(x, 0) = e^{ik\omega \cdot x} \chi_0\left(\frac{(x-x^{(0)}) \cdot \omega_\perp}{\delta_1}\right)$ when k is large. Therefore we constructed a solution $u(x, t)$ for $x \in \mathbf{R}^2 \setminus \Omega_1$, $t \in (0, T)$, $T = O(\frac{1}{k_3^\delta})$, k is large, such that

$$(3.24) \quad u(x, t) = e^{-i\frac{mk^2 t}{2h} - i\frac{mk}{h}x \cdot \omega} \chi_0\left(\frac{(x - x^{(0)}) \cdot \omega_\perp}{\delta_1}\right) \exp\left(i\frac{e}{hc} \int_0^\infty \omega \cdot A(x - s'\omega) ds'\right) + O(\varepsilon),$$

where ε can be chosen arbitrary small if k is large enough. \square

Let $x^{(0)} \in \mathbf{R}^2 \setminus \Omega_1$ and let ω and θ be two unit vectors (see Fig.1):

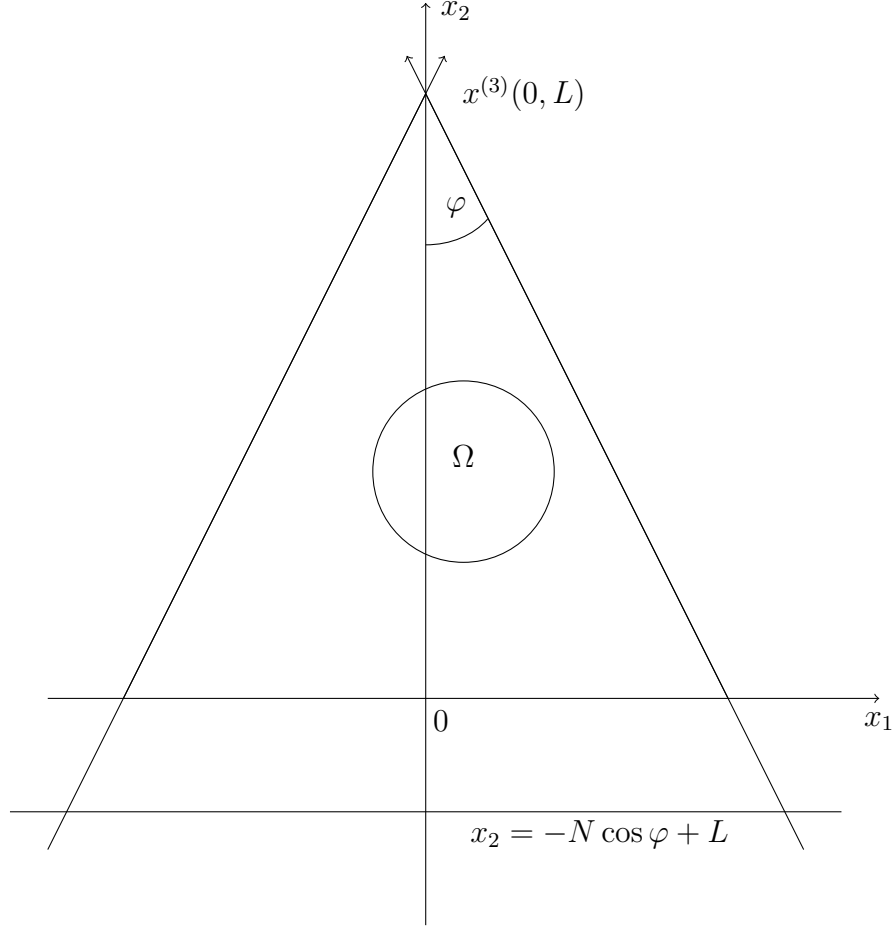


Figure 1.

Consider the difference of two solutions of the form (3.24) corresponding to $(x^{(0)}, \omega)$ and $(x^{(0)}, \theta)$, respectively:

$$(3.25) \quad u = v_1(x, t, \omega) - v_2(x, t, \theta),$$

$$(3.26)$$

$$v_1(x, t, \omega) = e^{-i\frac{mk^2t}{2h} + i\frac{mk}{h}x \cdot \omega} \chi_0\left(\frac{(x - x^{(0)}) \cdot \omega_{\perp}}{\delta_1}\right) \exp\left(i\frac{e}{hc} \int_0^{\infty} \omega \cdot A(x - s'\omega) ds'\right) + O(\varepsilon),$$

$$(3.27)$$

$$v_2(x, t, \theta) = e^{-i\frac{mk^2t}{2h} + i\frac{mk}{h}x \cdot \theta} \chi_0\left(\frac{(x - x^{(0)}) \cdot \theta_{\perp}}{\delta_1}\right) \exp\left(i\frac{e}{hc} \int_0^{\infty} \theta \cdot A(x - s'\theta) ds'\right) + O(\varepsilon),$$

where $\theta_{\perp} \cdot \theta = 0$. Note that modulo $O(\varepsilon)$ the support of v_1 is contained in a small neighborhood of the line $x = x^{(0)} + s\omega$ and the support of v_2 is contained in a small neighborhood of $x = x^{(0)} + s\theta$.

Let U_0 be a disk of radius ε_0 contained in $(\text{supp } v_1) \cap (\text{supp } v_2)$. We assume that $\chi_0\left(\frac{(x - x^{(0)}) \cdot \omega_{\perp}}{\delta_1}\right) = \chi_0\left(\frac{(x - x^{(0)}) \cdot \theta_{\perp}}{\delta_1}\right) = 1$ in U_0 . We have for $x \in U_0$ and $0 < t < T = \frac{1}{k^{\delta_3}}$

$$(3.28) \quad |v_1(x, t) - v_2(x, t)|^2 = \left|1 - e^{i\frac{mk}{h}x \cdot (\omega - \theta) + i(I_1 - I_2)}\right|^2 + O(\varepsilon) \\ = 4 \sin^2 \frac{1}{2} \left(\frac{mk}{h} x \cdot (\omega - \theta) + I_1 - I_2 \right) + O(\varepsilon), \\ I_1 = \frac{e}{hc} \int_0^{\infty} \omega \cdot A(x - s\omega) ds, \quad I_2 = \frac{e}{hc} \int_0^{\infty} \theta \cdot A(x - s\theta) ds,$$

and $k > k_0$, k_0 is large, $T \leq \frac{1}{k_0^{\delta_3}}$.

Choose $k_n > k_0$ such that

$$(3.29) \quad \frac{mk_n}{h} x^{(0)} \cdot (\omega - \theta) = 2\pi n, \quad n \in \mathbf{Z}.$$

Let, for simplicity, $\theta_1 = \omega_1$, $\theta_2 = -\omega_2$, $x^{(0)} = (0, L)$, $\tan \varphi = \frac{\theta_2}{\theta_1}$ is small. Define

$$I_{1N}(x, \omega) = \frac{e}{hc} \int_0^N \omega \cdot A(x^{(0)} - s\omega) ds,$$

$$I_{2N}(x, \theta) = \frac{e}{hc} \int_0^N \theta \cdot A(x^{(0)} - s\theta) ds,$$

$$I_{3N} = \frac{e}{hc} \int_{-N \sin \varphi}^{N \sin \varphi} A_1(s, -N \cos \varphi + L) ds$$

(see Fig.1). Note that $I_{1N}(x^{(0)}, \omega) - I_{2N}(x^{(0)}, \theta) + I_{3N} = \alpha$, where α is the magnetic flux (cf. (1.4)). We assume that

$$(3.30) \quad \alpha \neq 2\pi n, \quad \forall n \in \mathbf{Z}.$$

Since $|A| \leq \frac{C}{r}$, where r is the distance to Ω_1 , we have

$$(3.31) \quad |I_{3N}| \leq \frac{e}{hc} \frac{C}{N} 2N \sin \varphi = C_1 \frac{e}{hc} \sin \varphi.$$

When $N \rightarrow \infty$ we get

$$(3.32) \quad I_1 - I_2 = \alpha + O\left(\frac{e}{hc} \sin \varphi\right) \quad \text{for } x \in U_0.$$

Assuming that the radius of the disk U_0 is ε_0 we get

$$(3.33) \quad \left| \frac{mk_n}{h} (x - x^{(0)}) \cdot (\omega - \theta) \right| \leq C \frac{mk_n}{h} \varepsilon_0 \sin \varphi.$$

Therefore using (3.29), (3.31), (3.32), (3.33), fixing $k_n > k_0$ and choosing φ and ε_0 small enough we get

$$(3.34) \quad |v_1(x, t, \omega) - v_2(x, t, \theta)|^2 = 4 \sin^2 \frac{\alpha}{2} + O(\varepsilon).$$

Thus the probability density (3.34) depends on the magnetic flux α . Therefore the magnetic potentials belonging to different gauge equivalence classes make different physical impact. This proves the magnetic AB effect.

3.2 The three-dimensional case

The constructions of the subsection 3.1 can be carried out in the case of three dimensions. Consider, for example, a toroid Ω_1 in \mathbf{R}^3 as in Tonomura et al experiment (cf [T et al]). Let $x^{(0)}$ be a point outside of Ω_1 and let $\gamma_1 = \{x = x^{(0)} + s\omega, s \leq 0\}$ be a ray passing through the hole of the toroid. As in subsection 3.1 we can construct a solution $v_1(x, t, \omega)$ of the

form (3.26). In the case $n \geq 3$ dimensions there are $(n - 1)$ orthogonal unit vectors $\omega_{\perp j}, 1 \leq j \leq n - 1$, such that $\omega \cdot \omega_{\perp j} = 0, 1 \leq j \leq n - 1$, and we have to replace $\chi_0(\frac{(x-x^{(0)}) \cdot \omega_{\perp}}{\delta_1})$ in (3.26) by $\prod_{j=1}^{n-1} \chi_0(\frac{(x-x^{(0)}) \cdot \omega_{\perp j}}{\delta_1})$. Let $\gamma_2 = \{x = x^{(0)} + s\theta, s \leq 0\}$ be a ray passing outside of toroid and let $v_2(x, t, \omega)$ be the corresponding solution of the form (3.27). As in subsection 3.1 we get

$$|v_1(x, t, \omega) - v_2(x, t, \theta)|^2 = 4 \sin^2 \frac{\alpha}{2} + O(\varepsilon),$$

where $\alpha = \int_{\gamma} A(x) \cdot dx$, γ is a closed simple curve encircling Ω_1 and we assume that the angle between ω and θ is small.

Assuming that $\alpha \neq 2\pi n, \forall n \in \mathbf{Z}$, we obtain that the probability density $|v_1 - v_2|^2$ depends on α and this proves the AB effect.

3.3 The case of several obstacles

Let $\Omega_j, 1 \leq j \leq m, m > 1$, be obstacles in \mathbf{R}^2 , and let $\alpha_j = \frac{e}{hc} \int_{\gamma_j} A(x) \cdot dx$ be the magnetic fluxes generated by magnetic fields shielded in $\Omega_j, 1 \leq j \leq m$. Suppose that some α_j satisfy the condition (3.30). If the obstacles are close to each other it is impossible to repeat the construction of the subsection 3.1 separately for each Ω_j . Note that if the total flux $\sum_{j=1}^m \alpha_j = 2\pi p, p \in \mathbf{Z}$, then treating $\Omega = \cup_{j=1}^m \Omega_j$ as one obstacle we will miss the magnetic AB effect.

In this subsection we show how to determine all $\alpha_j \pmod{2\pi p}, 1 \leq j \leq m$, using the broken rays.

We shall introduce some notations.

Let $x^{(1)} \notin \Omega = \cup_{j=1}^m \Omega_j$. Denote by $\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_r$ the broken ray starting at $x^{(1)}$ and reflecting at Ω at points $x^{(2)}, \dots, x^{(r)}$. The last leg γ_r can be extended to the infinity. Denote by $\omega_p, 1 \leq p \leq r$, the directions of γ_p . The equations of $\gamma_1, \dots, \gamma_r$ are $x = x^{(1)} + s\omega_1, s_1 = 0 \leq s \leq s_2, x = x^{(2)} + s\omega_2, s_2 \leq s \leq s_3, \dots, x = x^{(r)} + s\omega_r, s_r \leq s < +\infty$. Here s_p are such that $x(s_p) = x^{(p)}, 1 \leq p \leq r$. Denote by $\tilde{\gamma} = \tilde{\gamma}_1 \cup \tilde{\gamma}_2 \cup \dots \cup \tilde{\gamma}_r$ the lifting of γ to $\mathbf{R}^2 \times (0, +\infty)$, where the equations of $\tilde{\gamma}_p$ are $x = x^{(p)} + s\omega_p, t = s, s_p \leq s \leq s_{p+1}, s_{r+1} = +\infty$. Note that the times when $\tilde{\gamma}$ hits the obstacles are $t_p = s_p, 2 \leq p \leq r$.

Let V_0 be a small neighborhood of $x^{(1)}$. Denote by $\gamma_y = \cup_{p=1}^r \gamma_{py}$ the broken ray that starts at $y \in V_0$ at $t = 0$. We assume that γ_{1y} has the form $x = y + s\omega_1, 0 \leq s \leq s_2(y)$, where $x^{(2)}(y) = y + s_2(y)\omega_1$ is the point where

γ_{1y} hits $\partial\Omega$. In particular, $\gamma_{x^{(1)}} = \gamma$. Let $U_0(t) = \{x = x(t)\}$ be the set of endpoints at the time t of $\tilde{\gamma}_y$, $y \in V_0$. Note that there is a one-to-one correspondence between $y \in V_0$ and $x(t) \in U_0(t)$. Therefore we shall denote the broken ray starting at $y \in V_0$ and ending at $x(t)$ at the time t by $\gamma(x(t))$ instead of γ_y . As in [E3], [E4] we can construct a geometric optics solution of $(\frac{h^2}{2m} \frac{\partial^2}{\partial t^2} + H)w_N = 0$ in $(\mathbf{R}^2 \setminus \Omega) \times (0, +\infty)$ in the form

$$(3.35) \quad w_N(x, t) = \sum_{p=1}^r \sum_{n=0}^N e^{i\frac{mk}{h}(\psi_p(x)-t)} \frac{a_{pn}(x, t)}{(ik)^n} + \sum_{p=1}^r \sum_{n=0}^N e^{i\frac{mk}{h}(\psi_p(x)+t)} \frac{b_{pn}(x, t)}{(ik)^n},$$

where

$$(3.36) \quad |\nabla \psi_p(x)| = 1, \quad \frac{\partial \psi_p(x^{(p)})}{\partial x} = \omega_p, \quad 1 \leq p \leq r, \\ \psi_1(x) = x \cdot \omega_1.$$

We have that $a_{pn}(x, t) = b_{pn}(x, -t)$ and $a_{pn}(x, t)$ satisfy the transport equations

$$(3.37) \quad \frac{\partial a_{pn}}{\partial t} + \frac{\partial \psi_p(x)}{\partial x} \cdot \frac{\partial a_{pn}}{\partial x} + \frac{1}{2} \Delta \psi_p a_{pn} - i \frac{e}{hc} A(x) \cdot \frac{\partial \psi_p}{\partial x} a_{pn} = f_{pn}(x, t), \\ 1 \leq p \leq r, \quad 0 \leq n \leq N,$$

where $f_{p0} = 0$, f_{pn} depend on a_{pj} for $n \geq 1$, $0 \leq j \leq n-1$. The following boundary conditions hold on $\partial\Omega \times (0, +\infty)$:

$$(3.38) \quad \psi_p|_{\partial\Omega \times (0, +\infty)} = \psi_{p+1}|_{\partial\Omega \times (0, +\infty)}, \quad 1 \leq p \leq r-1, \\ a_{pn}|_{\partial\Omega \times (0, +\infty)} = -a_{p+1, n}|_{\partial\Omega \times (0, +\infty)}, \quad 1 \leq p \leq r-1.$$

Conditions (3.38) imply that

$$w_N|_{\partial\Omega \times (0, +\infty)} = 0.$$

We impose the following initial conditions:

$$(3.39) \quad a_{10}(x, 0) = \frac{1}{2} \chi_0 \left(\frac{(x - x^{(1)}) \cdot \omega_{1\perp}}{\delta_1} \right) \chi_0 \left(\frac{(x - x^{(1)}) \cdot \omega}{\delta_2} \right), \\ a_{1n}(x, 0) = 0, \quad n \geq 1.$$

We assume that δ_1, δ_2 in (3.37) are small, so that the support of the first sum in (3.35) is contained in a small neighborhood of $\tilde{\gamma} = \cup_{p=1}^r \tilde{\gamma}_p$. We define $a_{pn}(x, t)$ as zero outside of this neighborhood of $\tilde{\gamma}$.

Let $x^{(0)} \in \gamma_r$ and $(x^{(0)}, t^{(0)})$ be a corresponding point on $\tilde{\gamma}_r$. It was shown in [E3], [E4] that

$$(3.40) \quad a_{r0}(x, t) = c_0(x, t) \exp \left(\frac{ie}{\hbar c} \int_{\gamma(x, t)} A(x) \cdot dx \right) + O\left(\frac{1}{k}\right),$$

where $\gamma(x, t)$ is the broken ray starting in a neighborhood of $x^{(1)}$ at $t = 0$ and ending at (x, t) , $c(x, t) \neq 0$ on $\gamma(x, t)$.

As in subsection 3.1 we have that $a_{rn}(x, t)$, $n \geq 1$ satisfy the estimates of the form (3.5).

Let $\tilde{\gamma}(x^{(0)}, t^{(0)})$ be the broken ray starting at $(x^{(1)}, 0)$ and ending at $(x^{(0)}, t^{(0)})$, where $x^{(0)} \in \gamma_r$. Let

$$(3.41) \quad u_N(x, t) = \frac{e^{-i\frac{\pi}{4}} \sqrt{m}}{\sqrt{2\pi\hbar t}} \int_{-\infty}^{\infty} e^{\frac{imx_0^2}{2\hbar t}} w_N(x, x_0) dx_0$$

where $w_N(x, x_0)$ is the same as in (3.35). We assume in this subsection that

$$(3.42) \quad t = \frac{t'}{k}, \quad 0 \leq t' \leq T'.$$

Applying the stationary phase method to (3.41) and using (3.40), (3.42) we get for x belonging to a neighborhood of $x^{(0)}$

$$(3.43) \quad \begin{aligned} u_N(x, t) &= \exp \left(i \left(-\frac{mk^2 t}{2\hbar} + \frac{mk}{\hbar} \psi_r(x) \right) \right) c_0(x, kt) \exp \left(\frac{ie}{\hbar c} \int_{\gamma(x, t')} A(x) \cdot dx \right) \\ &\quad + O\left(\frac{1}{k}\right), \end{aligned}$$

where $t' = kt$, t' belongs to a neighborhood of $t^{(0)}$.

Analogously to subsection 3.1 we get that there exists $R_N(x, t)$ such that $R_N(x, t) = O\left(\frac{1}{k}\right)$, $t = \frac{t'}{k}$, $0 \leq t' \leq T'$, and

$$(3.44) \quad u(x, t) = u_N(x, t) + R_N(x, t)$$

is the exact solution of (1.1) with the boundary conditions $u|_{\partial\Omega \times (0, \frac{T'}{k})} = 0$ and the initial condition

$$(3.45) \quad u(x, 0) = \chi_0\left(\frac{(x - x^{(1)}) \cdot \omega_{1\perp}}{\delta_1}\right) \chi_0\left(\frac{(x - x^{(1)}) \cdot \omega_1}{\delta_2}\right).$$

□

Denote by $\tilde{\beta} = \{x = x^{(2)} + s\theta, t = s, 0 \leq s \leq t^{(0)}\}$ the ray starting at $(x^{(2)}, 0)$ and ending exactly at the point $(x^{(0)}, t^{(0)})$. Analogously to (3.27) we can construct a solution $v(x, t)$ of (1.1), satisfying (1.2) and such that

$$\begin{aligned} v(x, t) = & \chi_0\left(\frac{(x - x^{(2)}) \cdot \theta_{\perp}}{\delta_1}\right) \chi_0\left(\frac{(x - x^{(2)}) \cdot \theta - kt}{\delta_2}\right) c_1(x, kt) \\ & \cdot \exp\left(-i\frac{mk^2t}{2h} + i\frac{mk}{h}\theta + \frac{ie}{hc} \int_0^{t'} A(x^{(2)} + s\theta) \cdot \theta dx\right) + O\left(\frac{1}{k}\right), \end{aligned}$$

where $t = \frac{t'}{k}$, $(x, t') \in U_0$, where U_0 is a neighborhood of $(x^{(0)}, t^{(0)})$.

We choose initial conditions $c_1(x, 0)$ such that (cf. (3.43))

$$c_1(x^{(0)}, t^{(0)}) = c(x^{(0)}, t^{(0)}).$$

Note that

$$\int_0^{t'} \theta \cdot A(x^{(2)} + s\theta) \cdot \theta ds = \int_{\beta(x, t')} A \cdot dx.$$

As in (3.28) we have near $(x^{(0)}, \frac{t^{(0)}}{k})$

$$|u(x, t) - v(x, t)|^2 = |c(x^{(0)}, t^{(0)})|^2 4 \sin^2 \frac{1}{2} \left(\frac{mk}{h} (\psi_r(x) - \theta \cdot x) + I_1 - I_2 \right) + O(\varepsilon),$$

where

$$I_1 = \frac{e}{hc} \int_{\gamma(x^{(0)}, t^{(0)})} A \cdot dx, \quad I_2 = \frac{e}{hc} \int_{\beta(x^{(0)}, t^{(0)})} A \cdot dx.$$

Choose $k_n > k_0$ such that

$$\frac{mk_n}{h} (\psi_r(x^{(0)}, t^{(0)}) - x^{(0)} \cdot \theta) = 2\pi n, \quad n \in \mathbf{Z},$$

and choose the initial points $x^{(1)}$ and γ_1 and $x^{(2)}$ on β far enough from Ω to have the integral

$$I_3 = \frac{e}{hc} \int_{\sigma} A \cdot dx$$

small. Here σ is the straight line connecting $x^{(1)}$ and $x^{(2)}$ and not intersecting Ω . Then if the neighborhood U_0 is small enough we get

$$|u(x, t) - v(x, t)|^2 = |c(x^{(0)}, t^{(0)})| 4 \sin^2 \frac{\alpha}{2} + O(\varepsilon),$$

where $\alpha = I_1 - I_2 + I_3$, $t = \frac{t'}{k}$, $(x^{(0)}, t') \in U_0$. Note that α is the sum of magnetic fluxes of obstacles that are bounded by $\gamma \cup (-\beta) \cup \sigma$. Varying γ and β at least m times we get enough linear relations to recover all $\alpha_j \pmod{2\pi n}$, $1 \leq j \leq m$.

Remark 3.1. Even in the case of one obstacle it is sometimes convenient to consider broken rays reflecting from the artificial boundaries (mirrors). Note that mirrors were used in the original AB experiment.

4 The proof of the electric Aharonov-Bohm effect

Let D be a domain in $\mathbf{R}^n \times [0, T]$ and let $D_{t_0} = D \cap \{t = t_0\}$. Assume that D_{t_0} depends continuously on $t_0 \in [0, T]$ and that normals to $D \setminus (\overline{D_0} \cup \overline{D_T})$ are not parallel to the t -axis. Suppose that the magnetic potential $A(x, t) = 0$ in D and consider the Schrödinger equation in D :

$$(4.1) \quad ih \frac{\partial u(x, t)}{\partial t} + \frac{h^2}{2m} \Delta u(x, t) - eV(x, t)u(x, t) = 0, \quad 0 < t < T,$$

with zero Dirichlet boundary condition

$$(4.2) \quad u|_{\partial D_t} = 0 \quad \text{for } 0 < t < T$$

and nonzero initial condition

$$(4.3) \quad u(x, 0) = u_0(x), \quad x \in D_0.$$

Suppose that electric field $E = \frac{\partial V}{\partial x} = 0$ in D . If D_t is connected for all $t \in [0, T]$ then $V(x, t) = V(t)$, i.e. $V(t)$ does not depend on x . Making the gauge transformation

$$(4.4) \quad v(x, t) = \exp\left(i \frac{e}{h} \int_0^t V(t') dt'\right) u(x, t)$$

we get that $v(x, t)$ satisfies the Schrödinger equation

$$(4.5) \quad ih \frac{\partial v}{\partial t} + \frac{h^2}{2m} \Delta v(x, t) = 0,$$

where

$$(4.6) \quad v|_{\partial D_t} = 0 \quad \text{for } 0 < t < T,$$

$$(4.7) \quad v(x, 0) = u_0(x), \quad x \in D_0.$$

Therefore $V(t)$ is gauge equivalent to zero electric potential, i.e. there is no electric AB effect in the case when D_{t_0} are connected for all $t_0 \in [0, T]$. To have the electric AB effect the domain D must have a more complicated topology. \square

We shall describe an electric AB effect when $A = 0, E = 0$ in D but the electric potential $V(x, t)$ is not gauge equivalent to the zero potential.

Consider the cylinder $D^{(1)} = \Omega \times [-T - 1, T + 1]$ where Ω is the unit disk $x_1^2 + x_2^2 < 1$. Let $D_{t_0}^{(1)} = D^{(1)} \cap \{t = t_0\}$. For any $t_0 \in [-T, T]$ denote by $D_{t_0}^{(2)}$ the part of $D_{t_0}^{(1)}$ where $|x_1| < \frac{1}{2} - \frac{1}{2} \frac{t_0^2}{T^2}$. Let $D^{(2)}$ be the union of $D_{t_0}^{(2)}$ when $-T \leq t_0 \leq T$ and let $D = D^{(1)} \setminus \overline{D}^{(2)}$. The domains D_{t_0} change the topology when t changes on $[-T - 1, T + 1]$ (see Fig. 2).

The slices $D_{t_0} = D \cap \{t = t_0\}$ are disks when $t_0 \in [-1 - T, -T)$ and $t_0 \in (T, T + 1]$ and they are not connected domains $D_{t_0} = D_{t_0}^{(1)} \setminus D_{t_0}^{(2)}$ when $t_0 \in (-T, T)$. If γ is a closed contour surrounding $D^{(2)}$ then γ is not homotopic to a point.

We consider the Schrödinger equation (4.1) in D with nonzero initial condition (4.3) for $t = -T - 1$ and the zero boundary condition (4.2) for $-T - 1 < t \leq T + 1$.

The gauge group $G(\overline{D})$ consists of all $g(x, t)$ in \overline{D} such that $|g(x, t)| = 1$. It follows from the topology of D that any $g(x, t) \in G(\overline{D})$ has the form:

$$g(x, t) = e^{in\theta + \frac{i}{h}\varphi(x, t)},$$

where $\varphi(x, t)$ is real-valued and differentiable in \overline{D} and θ is the polar angle in (x_1, t) -plane. If $V_1(x, t)$ and $V_2(x, t)$ are gauge equivalent and if γ is a closed contour in D encircling $D^{(2)}$, then

$$e \int_{\gamma} V_1(x, t) dt - e \int_{\gamma} V_2(x, t) dt = ih \int_{\gamma} g^{-1}(x, t) \frac{\partial g}{\partial t} dt = 2\pi h n, \quad n \in \mathbf{Z}.$$

Suppose $V(x, t) = 0$ for $t \in [-T-1, -T+\varepsilon]$ and $[T-\varepsilon, T+1]$, where $\varepsilon > 0$, and suppose $V(x, t) = V_1(t)$ when $t \in [-T+\varepsilon, T-\varepsilon]$, $x_1 > 0$, $V(x, t) = V_2(t)$ when $t \in [-T+\varepsilon, T-\varepsilon]$, $x_1 < 0$. Then $E = \frac{\partial V}{\partial x} = 0$ in D .

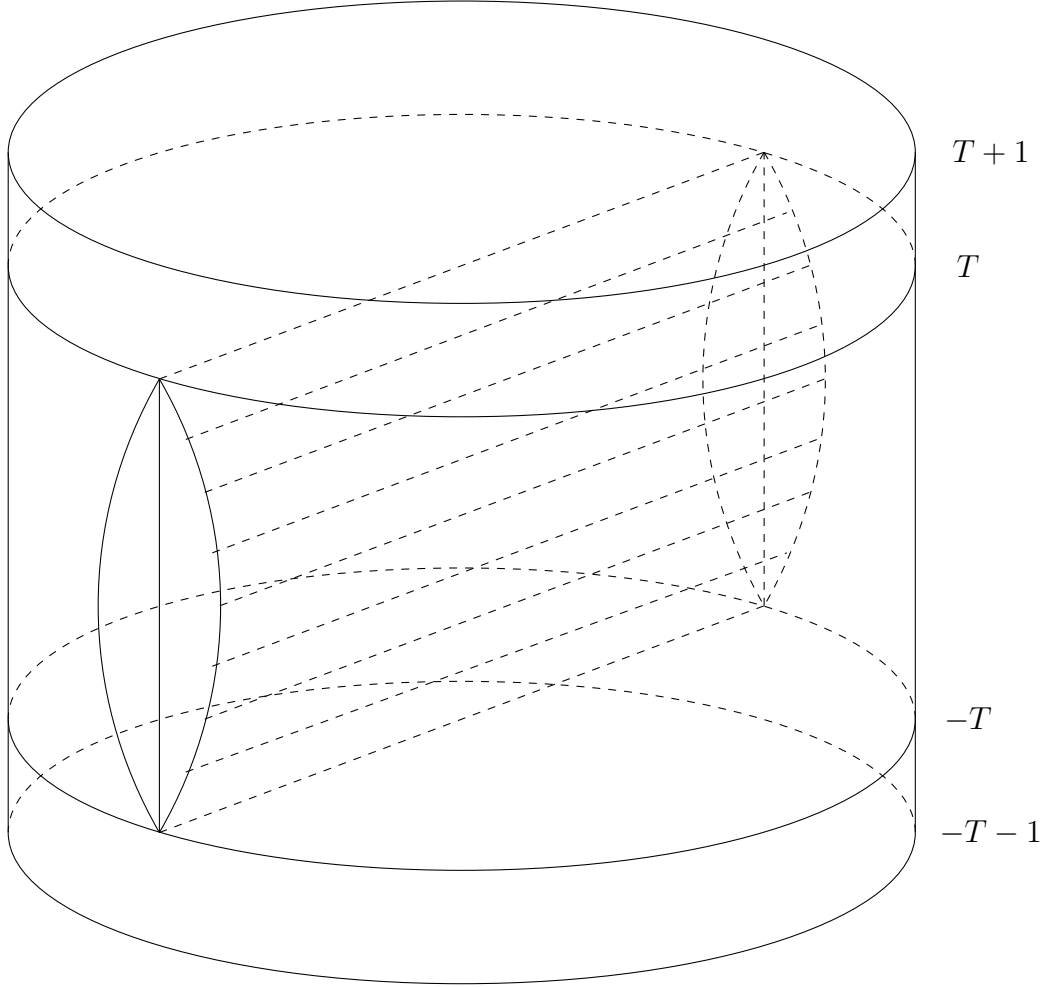


Figure 2.

Denote $\alpha_j = \frac{\varepsilon}{h} \int_{-T+\varepsilon}^{T-\varepsilon} V_j(t) dt$, $j = 1, 2$, and suppose $\alpha_1 - \alpha_2 \neq 2\pi n$, $\forall n \in \mathbf{Z}$. Since the electric flux $\alpha = \frac{\varepsilon}{h} \int_{\gamma} V(x, t) dt = \alpha_1 - \alpha_2 \neq 2\pi n$, $\forall n$ the electric potential $V(x, t)$ is not gauge equivalent to the zero potential.

Let $u(x, t)$ be the solution of (4.1) in D with the initial and boundary conditions (4.2), (4.3) and let $v(x, t)$ be the solution of (4.5) in D with $V(x, t) = 0$ and the same initial and boundary conditions as $u(x, t)$.

We shall show that if $\alpha = \frac{e}{h} \int_{\gamma} V(x, t) dt = \alpha_1 - \alpha_2 \neq 2\pi m, \forall m \in \mathbf{Z}$ then the probability densities $|u(x, t)|^2$ and $|v(x, t)|^2$ differ near $t = T$. Therefore $V(x, t)$ makes a physical impact different from the impact of the zero potential. This will prove the electric AB effect. Since $u(x, t)$ and $v(x, t)$ have the same initial and boundary conditions (4.2), (4.3) and $V(x, t) = 0$ for $t \in [-T - 1, -T + \varepsilon]$ we have that $u(x, t) = v(x, t)$ for $t \leq -T + \varepsilon$. Denote by Π_1 and Π_2 two connected components of $(D^{(1)} \setminus D^{(2)}) \cap (-T + \varepsilon, T - \varepsilon)$. Note that in $\Pi_i, i = 1, 2$, we have

$$u(x, t) = v(x, t) \exp \left(-i \frac{e}{h} \int_{-T+\varepsilon}^t V_i(t') dt' \right), \quad (x, t) \in \Pi_i.$$

Let $c(x, t) = 1$ for $t \leq -T + \varepsilon$, $c(x, t) = e^{-i \frac{e}{h} \int_{-T+\varepsilon}^t V_i(t') dt'}$ in Π_i for $t \in (-T + \varepsilon, T - \varepsilon), i = 1, 2$. Then $u(x, t) = c(x, t)v(x, t)$, i.e. $V(x, t)$ is gauge equivalent to zero in $D \cap (-T - 1, T - \varepsilon)$. However, $\lim_{t \rightarrow T} c(x, t)$ is equal to $\exp(-i \frac{e}{h} \int_{-T+\varepsilon}^T V_1(t') dt')$ for $x_1 > 0$ and is equal to $\exp(-i \frac{e}{h} \int_{-T+\varepsilon}^T V_2(t') dt')$ when $x_1 < 0$. Since $\exp(-i \frac{e}{h} \int_{-T+\varepsilon}^T V_1(t') dt') \neq \exp(-i \frac{e}{h} \int_{-T+\varepsilon}^T V_2(t') dt')$ we have that $c(x, t)$ is discontinuous at $x_1 = 0, t = T$. \square

Note that $u(x, t)$ and $v(x, t)$ satisfy the same equation (4.5) in $D \cap \{T \leq t \leq T + 1\}$ and

$$\begin{aligned} u(x, T) &= e^{i\alpha_1} v(x, T) \quad \text{for } x_1 > 0, \\ u(x, T) &= e^{i\alpha_2} v(x, T) \quad \text{for } x_1 < 0. \end{aligned}$$

We shall show that the probability densities $|u(x, t)|^2$ and $|v(x, t)|^2$ are not equal identically for $T < t < T + \varepsilon$. Therefore the physical impact of the electric potential $V(x, t)$ differs from the impact of the zero potential.

To prove that $|u(x, t)|^2 \neq |v(x, t)|^2$ in D we consider $w(x, t) = e^{-i\alpha_2} u(x, t)$. Then $w(x, T) = v(x, T)$ for $x_1 < 0, w(x, T) = e^{i(\alpha_1 - \alpha_2)} v(x, T)$ for $x_1 > 0$.

Proposition 4.1. *If $|w(x, t)|^2 = |v(x, t)|^2$ for $T \leq t \leq T + \varepsilon$ and $v(x, T) = w(x, T)$ for $x_1 < 0$, then $v(x, t) = w(x, t)$ for all $T < t < T + \varepsilon, x_1^2 + x_2^2 \leq 1$.*

Proof of Proposition 4.1. Let $v(x, t) \neq 0$ in some neighborhood O of $(x_1^{(0)}, 0, T), x_1^{(0)} < 0$. Denote $R(x, t) = |v(x, t)|, \Phi(x, t) = \arg v(x, t)$, i.e. $v(x, t) = R(x, t)e^{i\Phi(x, t)}$. Substituting in (4.5) and separating the real and the imaginary parts we get

$$(4.8) \quad -hR_t = \frac{h^2}{2m}(2\nabla R \cdot \nabla \Phi + R\Delta \Phi),$$

$$(4.9) \quad h\Phi_t R = \frac{h^2}{2m}(\Delta R - R|\nabla\Phi|^2).$$

Suppose R is given. Then (4.9) is a first order partial differential equation in Φ and therefore the initial data $\Phi(x, T)$ near $(x_1^{(0)}, 0)$ uniquely determines $\Phi(x, t)$ in the neighborhood O . Let $w = R_1(x, t)e^{i\Phi_1(x, t)}$. Note that $w(x, t)$ also satisfies (4.5) for $t > T$ and Φ_1 satisfies (4.9). Since $R = R_1$ in O and, since $\Phi_1(x_1, 0, T) = \Phi(x_1, 0, T)$ in O , we have that $w(x, t) = v(x, t)$ in O . Then $w(x, t) = v(x, t)$ for $T < t < T + \varepsilon$, $x_1^2 + x_2^2 < 1$, by the unique continuation property (see [I], section 6). By the continuity in t we get $v(x, T) = w(x, T)$ for $x_1 > 0$ and this is a contradiction with $w(x, T) = e^{i(\alpha_1 - \alpha_2)}v(x, T)$ for $x_1 > 0$. Therefore $|u(x, t)|^2 \not\equiv |v(x, t)|^2$ for $T < t < T + \varepsilon$, i.e. the AB effect holds. \square

Remark 4.1. In the proof of Proposition 4.1 we used that $R(x, t) = |v(x, t)| \not\equiv 0$ for $t = T$ and $x_1 < 0$ and $|v(x, t)| \not\equiv 0$ for $t = T$ and $x_1 > 0$. We shall show that this can be achieved by the appropriate choice of the initial condition $u_0(x)$ in (4.7). Choose any $v(x, T)$ such that $v(x, T) \not\equiv 0$ for $x_1 > 0$ and $v(x, T) \equiv 0$ for $x_1 < 0$.

Solve the backward initial value problem for (4.5) with the boundary condition (4.6) and the initial condition $v(x, T)$ for $t = T$. Then we take $v(x, -T - 1)$ as the initial condition $u_0(x)$ in (4.7) and (4.3).

4.1 A possible physical experiment to demonstrate the electric AB effect

The class of domains D with nontrivial topology that leads to the electric AB effect is large. Below we give an example that can lead to a physical experiment to demonstrate the electric AB effect.

Denote by $\Omega(\tau)$ the interior of the unit disk $x_1^2 + x_2^2 \leq 1$ with removed two parts $\Delta(\tau)$ and $\Delta(-\tau)$ depending on the parameter τ , $0 \leq \tau \leq \frac{1}{2}$ (see Fig. 3).

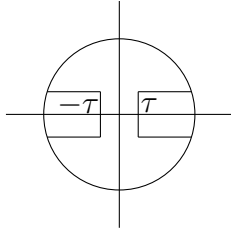


Fig.3

Let D be the following domain in $\mathbf{R}^2 \times [0, T + 1]$:

$$D_t = \Omega\left(\frac{1}{2} - t\right) \text{ for } 0 \leq t \leq \frac{1}{2},$$

$$D_t = \Omega(0) \text{ for } \frac{1}{2} \leq t \leq T + \frac{1}{2},$$

$$D_t = \Omega\left(t - \frac{1}{2} - T\right) \text{ for } T + \frac{1}{2} \leq t \leq T + 1.$$

Here $D_{t_0} = D \cap \{t = t_0\}$.

Therefore $\Delta(\tau)$ and $\Delta(-\tau)$ increase in size from $\tau = \frac{1}{2}$ to $\tau = 0$ when $0 \leq t \leq \frac{1}{2}$. Then they do not move for $\frac{1}{2} \leq t \leq T + \frac{1}{2}$. Note that D_t consists of the components D_t^+ and D_t^- for $\frac{1}{2} \leq t \leq \frac{1}{2} + T$. When $T + \frac{1}{2} \leq t \leq T + 1$ the parts $\Delta(\tau)$ and $\Delta(-\tau)$ return back to the initial position $\tau = \frac{1}{2}$.

Such moving domain is easy to realize experimentally. We can arrange that $V(t) = 0$ in D for $0 \leq t \leq \frac{1}{2} + \varepsilon$ and for $\frac{1}{2} + T - \varepsilon < t \leq T + 1$, $V(t) = V_1(t)$ in D_t^+ , $V(t) = V_2(t)$ in D_t^- for $\frac{1}{2} \leq t \leq T + \frac{1}{2}$. Suppose

$$\frac{e}{h} \int_{\frac{1}{2}}^{T+\frac{1}{2}} V_1(t) dt - \frac{e}{h} \int_{\frac{1}{2}}^{T+\frac{1}{2}} V_2(t) dt \neq 2\pi n, \quad \forall n \in \mathbf{Z}.$$

Then one can show theoretically as in Proposition 4.1 that the electric AB effect holds for $\frac{1}{2} + T < t < 1 + T$. Hopefully this can be shown also experimentally by measuring $|u(x, t)|^2$ for $\frac{1}{2} + T < t < 1 + T$ and comparing these measurements with the measurements for the zero potential in D .

We assume that the initial data $u_0(x)$ for $V(t)$ and $V \equiv 0$ are the same.

5 Combined electric and magnetic AB effect

5.1 Class of domains

Consider the following domain $D \subset \mathbf{R}^n \times (0, T)$: Let $T_0 = 0 < T_1 < \dots < T_r = T$. Denote by D_{t_0} the intersection of D with the plane $t = t_0$. Then for $t_0 \in (T_{p-1}, T_p)$, $p = 1, \dots, r$, we have $D_{t_0} = \Omega_0 \setminus \overline{\Omega_p(t_0)}$, where Ω_0 is a simply-connected domain in \mathbf{R}^n , $\Omega_p(t_0) = \cup_{j=1}^{m_p} \Omega_{pj}(t_0)$, $\overline{\Omega_{pj}(t_0)} \cap \overline{\Omega_{pk}(t_0)} = \emptyset$ for $j \neq k$, $\overline{\Omega_{pj}(t_0)} \subset \Omega_0$, $\Omega_{pj}(t_0)$ are smooth domains (obstacles). Note that m_p may be different for $p = 1, \dots, r$. We assume that $\Omega_p(t_0)$ depends smoothly on $t_0 \in (T_{p-1}, T_p)$. Also we assume that D_{t_0} depends continuously on $t_0 \in [0, T]$.

Note that some obstacles may merge or split when t_0 crosses $T_p, p = 1, \dots, r-1$ (see Fig. 4).

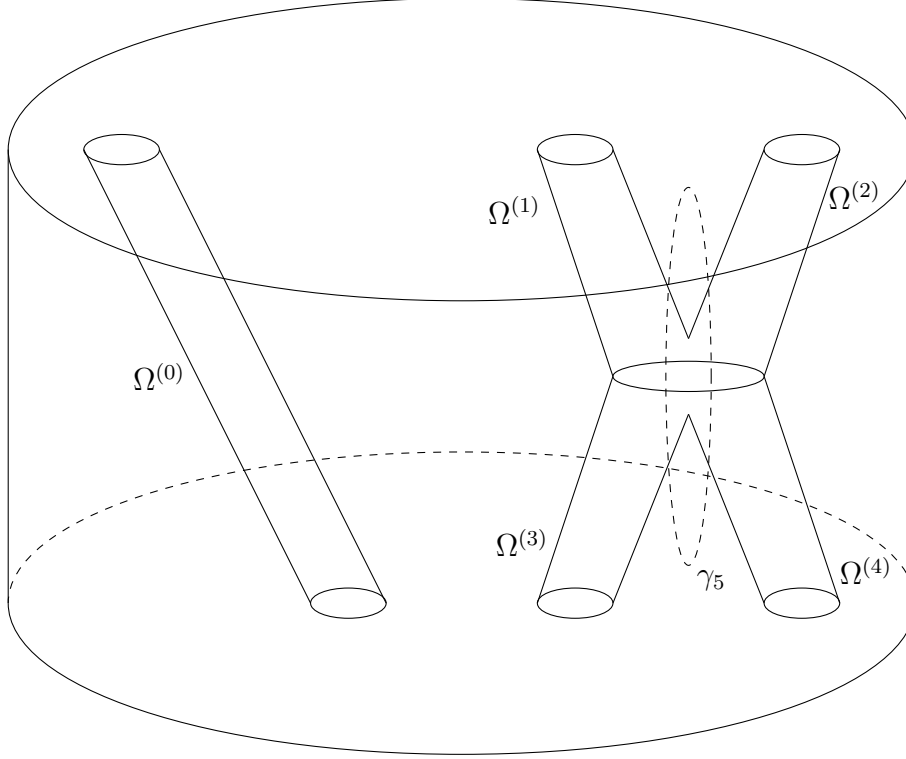


Figure 4.

We shall study the time-dependent Schrödinger equation in D :

$$(5.1) \quad ih \frac{\partial u(x, t)}{\partial t} - \frac{1}{2m} \sum_{j=1}^n \left(-ih \frac{\partial}{\partial x_j} - \frac{e}{c} A_j(x, t) \right)^2 u - eV(x, t)u(x, t) = 0$$

with smooth time-dependent magnetic potential $A(x, t) = (A_1(x, t), \dots, A_n(x, t))$ and electric potential $V(x, t)$, $(x, t) \in D$.

We assume that the normals to $\partial\Omega_p(t)$ are not parallel to the t -axis for any $t \in [0, T]$.

The gauge group $G(\overline{D})$ is the group of all $C^\infty(\overline{D})$ complex-valued functions $g(x, t)$ such that $|g(x, t)| = 1$ in \overline{D} (cf. §1).

Electromagnetic potentials $(A(x, t), V(x, t))$ and $(A'(x, t), V'(x, t))$ are called

gauge equivalent is there exists $g(x, t) \in G(\overline{D})$ such that

$$(5.2) \quad \begin{aligned} \frac{e}{c}A'(x, t) &= \frac{e}{c}A(x, t) + ihg^{-1}(x, t)\frac{\partial g}{\partial x} \\ eV'(x, t) &= eV(x, t) - ihg^{-1}(x, t)\frac{\partial g}{\partial t}. \end{aligned}$$

We shall consider the case when the magnetic and the electric fields are zero in D , i.e., $B = \text{curl } A(x, t) = 0$, $E = -\frac{1}{c}\frac{\partial A(x, t)}{\partial t} - \frac{\partial V(x, t)}{\partial x} = 0$, $(x, t) \in D$. In this case the integral

$$(5.3) \quad \alpha = \frac{e}{h} \int_{\gamma} \frac{1}{c} A(x, t) \cdot dx - V(x, t) dt$$

over a closed curve γ in D does not change if we deform γ continuously in D .

The integral (5.3) is called the electromagnetic flux. It is easy to describe all gauge equivalent classes of electromagnetic potentials using the electromagnetic fluxes and assuming $B = E = 0$ in D .

Let $\gamma_1, \gamma_2, \dots, \gamma_l$ be a basis of the homology group of D , i.e., any closed contour γ in D is homotopic to a linear combination of $\gamma_1, \dots, \gamma_l$ with integer coefficients. Then fluxes

$$\alpha_j = \frac{e}{h} \int_{\gamma_j} \frac{1}{c} A(x, t) \cdot dx - V(x, t) dt, \quad 1 \leq j \leq l,$$

modulo $2\pi n, n \in \mathbf{Z}$, determine a gauge equivalent class of $(A(x, t), V(x, t))$, i.e. $(A(x, t), V(x, t))$ and $(A'(x, t), V'(x, t))$ are gauge equivalent iff $\alpha_j - \alpha'_j = 2\pi m_j$, $m_j \in \mathbf{Z}$, $1 \leq j \leq l$, where $\alpha'_j = \frac{e}{h} \int_{\gamma_j} \frac{1}{c} A' \cdot dx - V'(x, t) dt$.

In the next section we shall prove that the electromagnetic potentials belonging to different gauge equivalent classes have a different physical impact, for example, the probability density $|u(x, t)|^2$ will be different for some $u(x, t)$.

5.2 The proof of the electromagnetic AB effect

We shall introduce localized geometric optics type solutions $u(x, t)$ of the Schrödinger equation (5.1) in D depending on a large parameter k and satisfying the zero initial conditions

$$(5.4) \quad u(x, 0) = 0, \quad x \in D_0,$$

and zero boundary conditions on the boundaries of obstacles

$$(5.5) \quad u(x, t)|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbf{R}^n \times (0, T)$ is the union of all obstacles $\Omega_p(t)$, $t \in [T_{p-1}, T_p]$, $p = 1, \dots, r$, and $D_0 = \Omega_0 \setminus \Omega_1(0)$, $\Omega_1(0) = \Omega \cap \{t = 0\}$. Such solutions were constructed in [E1]. Suppose $t_0 \in (T_{p-1}, T_p)$, $1 \leq p \leq r$. Suppose $\gamma(x^{(1)}, t_0) = \beta_1(t_0) \cup \dots \cup \beta_{d-1}(t_0) \cup \beta_d(x^{(1)}, t_0)$ is a broken ray in D_{t_0} with legs $\beta_1(t_0), \dots, \beta_d(x^{(1)}, t_0)$ reflecting at $\partial\Omega_p(t_0)$, starting at point $x^{(0)} \in \partial\Omega_0$ and ending at $x^{(1)} \in D_{t_0}$.

As in [E1] we can construct an asymptotic solution as $k \rightarrow \infty$ of the form (cf subsection 3.3):

$$(5.6) \quad u_N(x, t) = \sum_{j=1}^d e^{-\frac{imk^2}{2h}t + i\frac{mk}{h}\psi_j(x, t)} \sum_{n=0}^N \frac{a_{nj}(x, t, \omega)}{(ik)^n},$$

where $\psi_1(x, t) = x \cdot \omega$ and $\text{supp } u_N(x, t, \omega)$ is contained in a small neighborhood of $x = \gamma(x^{(1)}, t_0)$, $t = t_0$ (see [E1] for the details). As it was shown in [E1] one can find $u^{(N)}(x, t)$ such that $Lu^{(N)} = -Lu_N = O(\frac{1}{k^{N+1}})$ in D , $u^{(N)}|_{t=0} = 0$, and $u^{(N)}|_{\partial\Omega} = 0$, $u^{(N)}|_{\partial\Omega_0 \times (0, T)} = 0$ and such that $u^{(N)} = O(\frac{1}{k^{N-2}})$. Here L is the right hand side of (5.1). Then

$$(5.7) \quad u = u_N + u^{(N)}$$

is the exact solution of $Lu = 0$ in D , $u|_{t=0} = 0$, $x \in D_0$, $u|_{\partial\Omega} = 0$ for all $0 < t < T$.

Let $t_0 \in (T_p, T_{p+1})$ and let m_p be the number of the obstacles in D_{t_0} . It was proven in [E1], [E3] that $u(x, t)$ has the following form in the neighborhood U_0 of $(x^{(1)}, t_0)$:

$$(5.8) \quad u(x, t) = c(x, t) \exp \left(-i\frac{mk^2t}{2h} + i\frac{mk}{h}\psi_d(x, t) + \frac{ie}{hc} \int_{\gamma(x, t)} A(x, t) \cdot dx \right) + O\left(\frac{1}{k}\right),$$

Here $c(x^{(1)}, t_0) \neq 0$ and $\gamma(x, t)$ is a broken ray in D_t that starts at (y, t) , (y, t) is close to $(x^{(0)}, t_0)$, and such that the first leg of $\gamma(x, t)$ has the same direction as $\beta_1(t_0)$.

Note the difference between asymptotic solution (3.35) for the wave equations and asymptotic solution (5.8) for the Schrödinger equations. Solution (3.35) corresponds to the broken ray $\tilde{\gamma} = \cup_{j=1}^r \tilde{\gamma}_j$ in $\mathbf{R}^2 \times (0, +\infty)$ and solution (5.8) corresponds to the broken ray $\cup_{j=1}^d \beta_j$ in the plane $t = t_0$. \square

Let γ_1 be the ray $x = x^{(0)} + s\theta$, $s \geq 0$, $t = t_0$, starting at $(x^{(0)}, t_0)$ and ending at $(x^{(1)}, t_0)$. Choose $x^{(1)} \in \Omega_0$ such that γ_1 does not intersect $\Omega(t_0)$. We assume that Ω_0 is large enough that such $x^{(1)}$ exists (see Fig.5):

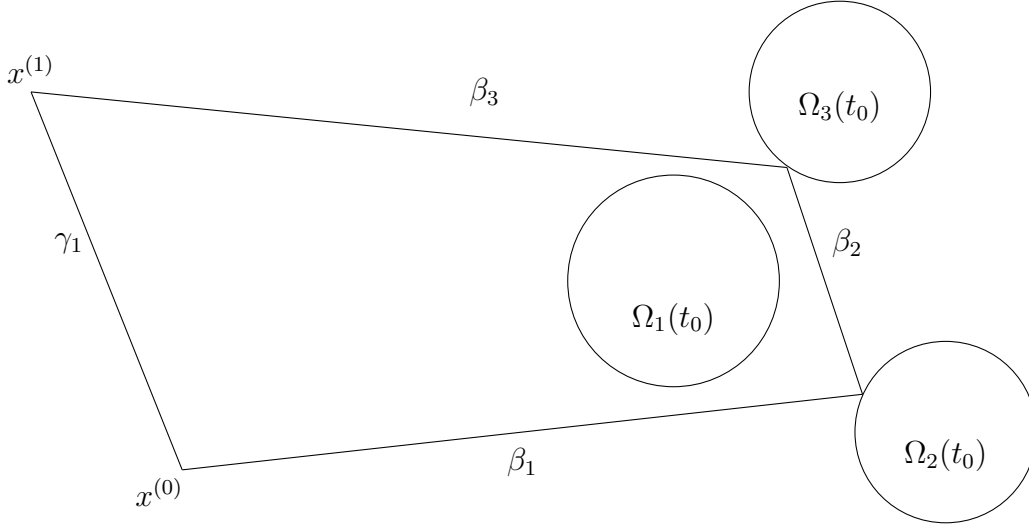


Fig.5

Let $v(x, t)$ be a geometric optic solution similar to (5.6) with $d = 1$ and corresponding to the ray γ_1 . We have, as in (5.8):

$$(5.9) \quad v(x, t) = c_1(x, t) \exp \left(-i \frac{mk^2 t}{2h} + i \frac{mk}{h} x \cdot \theta + \frac{ie}{hc} \int_{\gamma_1(x, t)} A(x, t) \cdot dx \right) + O\left(\frac{1}{k}\right).$$

We choose the initial value for $a_0(x, t, \theta)$ (cf. (5.6)) near $(x^{(0)}, t_0)$ such that

$$c_1(x^{(1)}, t_0) = c(x^{(1)}, t_0).$$

Consider $|u(x, t) - v(x, t)|^2$ in a neighborhood $\{(x, t) : |x - x^{(1)}| \leq \varepsilon_0, |t - t_0| < \varepsilon_0\}$.

As in subsection 3.3 we get for a small neighborhood of $(x^{(1)}, t_0)$

$$(5.10) \quad |u(x, t, \omega) - v(x, t, \theta)|^2 = |c(x^{(1)}, t_0)|^2 4 \sin^2 \frac{\alpha(t_0)}{2} + O(\varepsilon),$$

where

$$\alpha(t_0) = \frac{e}{hc} \left(\int_{\gamma(x^{(1)}, t_0)} A(x, t_0) \cdot dx - \int_{\gamma_1(x^{(1)}, t_0)} A(x, t_0) \cdot dx \right).$$

Note that $\alpha(t_0)$ is the sum of the fluxes of those obstacles $\Omega_{pj}(t_0)$, $1 \leq j \leq m_1$, that are encircled by $\gamma \cup \gamma_1$. As in subsection 3.3, varying γ and γ_1 at least m_p times we can recover (modulo $2\pi n$) $\alpha_{pj}(t_0)$, $1 \leq j \leq m_p$, where

$$(5.11) \quad \alpha_{pj}(t_0) = \frac{e}{hc} \int_{\gamma_{pj}(t_0)} A \cdot dx, \quad 1 \leq j \leq m_p,$$

and $\gamma_{pj}(t_0)$ is a simple contour in D_{t_0} encircling $\Omega_{pj}(t_0)$, $1 \leq j \leq m_p$. Note that α_{pj} are the same for any $t_0 \in (T_p, T_{p+1})$. We can repeat the same arguments for any $t_0 \neq T_1, \dots, T_{p-1}$.

Our class of time-dependent obstacles is such that D_{t_0} is connected for any $t_0 \in [0, T]$. It follows from this assumption that a basis of the homology group of D is contained in the set $\gamma_{pj}(t_p)$, $1 \leq j \leq m_p$, $t_p \in (T_{p-1}, T_p)$, $1 \leq p \leq r$.

Denote such basis by $\gamma^{(1)}(t^{(1)}), \dots, \gamma^{(l)}(t^{(l)})$. Then any closed contour γ in D is homotopic to a linear combination $\sum_{j=1}^l n_j \gamma^{(j)}(t^{(j)})$ where $n_j \in \mathbf{Z}$. Therefore the flux

$$(5.12) \quad \frac{e}{hc} \int_{\gamma} A \cdot dx - V dt = \sum_{j=1}^l n_j \alpha^{(j)}(t^{(j)}),$$

where $\alpha^{(j)}(t^{(j)}) = \frac{e}{hc} \int_{\gamma^{(j)}(t^{(j)})} A \cdot dx$.

Thus the fluxes $\alpha^{(j)}(t^{(j)})$, $1 \leq j \leq l$, mod $2\pi n$, $n \in \mathbf{Z}$, determine the gauge equivalence class of $A(x, t), V(x, t)$. Therefore computing the probability densities of appropriate solutions we are able to determine the gauge equivalence classes of electromagnetic potentials.

5.3 Example 5.1

Consider the domain shown in Fig. 4. Let γ_p , $0 \leq p \leq 4$, be simple closed curves encircling $\Omega^{(p)}$. There is also a simple closed curve γ_5 that is not homotopic to any closed curve contained in the plane $t = C$. Note that $\gamma_1 + \gamma_2 \approx \gamma_3 + \gamma_4$ where \approx means homotopic. Also $\gamma_5 \approx \gamma_1 - \gamma_3$. Therefore $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ is a basis of the homology group of D . We made an assumption that D_{t_0} is connected for any $t_0 \in [0, T]$. Under this assumption there is always a basis of the homology group consisting of "flat" closed curves, i.e. the curves containing in the planes $t = \text{const}$.

Let α_j be the fluxes corresponding to γ_j . Note that if γ is flat then $\alpha = \int_{\gamma} A \cdot dx$ is a magnetic flux. However $\alpha_5 = \frac{e}{h} \int_{\gamma_5} \frac{1}{c} A \cdot dx - V dt$ is an electro-

magnetic flux. Since $\gamma_5 \approx \gamma_1 - \gamma_3$ we have that $\alpha_5 = (\alpha_1 - \alpha_3) \pmod{2\pi n}$, $n \in \mathbf{Z}$. \square

Note that our approach allows to calculate first the magnetic fluxes. To determine electromagnetic flux we have to represent it as a linear combination of magnetic fluxes as in Example 5.1.

Remark 5.1. In this section we assumed that the domains D_t are connected for all $t \in [0, T]$. To incorporate the examples of §4 we have to allow D_t to be not connected for some $t \in [0, T]$, i.e. $D_t = \cup_{k=0}^m D_t^{(0)}$, where $D_t^{(0)}$ is the open component containing the neighborhood of $\partial\Omega_0$ and $D_t^{(k)}$, $1 \leq k \leq m$, are other open component that we assume to be simply connected.

Acting as above we can determine the magnetic fluxes α_j in $D_t^{(0)} \pmod{2\pi n}$. Since $D_t^{(k)}$, $k \geq 1$, are simply connected, we have that $\frac{e}{c}A = h \frac{\partial \varphi_k(x,t)}{\partial x}$ in $\overline{D_t^{(k)}}$. Making the gauge transformations with gauge $e^{i \frac{\varphi_k(x,t)}{h}}$ in $D_t^{(k)}$ we can get that $A = 0$ in $D_t^{(k)}$. Then we obtain the same situation as in the examples of §4. For example, in the case of Fig. 4 we can insert a domain of the form of Fig. 3 inside the tube $\Omega^{(0)}$. Note that in §4 we did not determine the electric flux as in §3, but only find whether the electric potentials are gauge equivalent or not.

5.4 Approximation of solutions in D by physically meaningful solutions

Let $u(x, t)$ be a solution of (5.1) in $(\mathbf{R}^n \times (0, T)) \setminus \Omega$, where $\Omega \subset \mathbf{R}^n \times (0, T)$ is the union of all obstacles, $0 \leq t \leq T$. We assume that

$$(5.13) \quad u|_{\partial\Omega} = 0$$

and

$$(5.14) \quad u(x, 0) = u_0(x), \quad x \in \mathbf{R}^n \setminus \Omega(0),$$

where $\Omega(t_0) = \Omega \cap \{t = t_0\}$, in particular, $\Omega(0) = \Omega \cap \{t = 0\}$.

Initial-boundary value problem (5.1), (5.13), (5.14) describes an electron confined to the region $\mathbf{R}^n \setminus \Omega(t_0)$, $0 < t_0 < T$. We shall assume that the initial data $u_0(x) \in H_2(\mathbf{R}^n \setminus \Omega(0)) \cap \mathring{H}_1(\mathbf{R}^n \setminus \Omega(0))$ (cf. [E1]). It follows from [E1] that $u(x, t) \in C((0, T), H_2(\mathbf{R}^n \setminus \Omega(t)) \cap \mathring{H}_1(\mathbf{R}^n \setminus \Omega(t)))$, i.e. $u(x, t)$ belongs to a space of continuous functions in t with values in $H_2(\mathbf{R}^n \setminus \Omega(t)) \cap \mathring{H}_1(\mathbf{R}^n \setminus$

$\Omega(t)$). Here $u(x, t) \in \mathring{H}_1(\mathbf{R}^n \setminus \Omega(t))$ means that $u(x, t) = 0$ on $\partial\Omega(t)$. We shall call such solutions $u(x, t)$ physically meaningful. In subsection 5.2 we considered solutions $v(x, t)$ of (5.1) defined in the domain D only with zero initial conditions in $\Omega_0 \setminus \Omega(0)$, zero boundary conditions on $\partial\Omega$ and having nonzero values on $\partial\Omega_0 \times (0, T)$. We assume that $u(x, t) \in C((0, T), H_2(\Omega_0 \setminus \Omega(t)))$. Then $u|_{\partial\Omega_0 \times (0, T)} = f$ where $f(x, t)$ is continuous in t , with values in $H_{\frac{3}{2}}(\partial\Omega_0)$. We shall denote such class of solutions by $W(D)$.

We shall show that any solution in $W(D)$ can be approximated by the restriction to D of physically meaningful solutions $u(x, t)$ such that $u(x, 0) = 0$ in $\Omega_0 \setminus \Omega(0)$, and this will make our proof of electromagnetic AB physically relevant.

Denote by V the Banach space of functions $u(x, t)$ in D with the norm $\|u\|_V = \int_0^T [u]_0 dt$, where $[v]_0$ is the L_2 -norm in $D_t = \Omega_0 \setminus \Omega(t)$. Let V^* be the dual space with the norm $\|v\|_{V^*} = \sup_{0 \leq t \leq T} [v]_0$. Denote by $K \subset V^*$ the closure in the V^* norm of the restrictions to D of all physically meaningful solutions such that $v(x, 0) = 0$ for $x \in D_0 = \Omega_0 \setminus \Omega(0)$.

Let K^\perp be the set of all $v \in V$ such that $(u, v) = 0$ for all $u \in K$. Here (u, v) is the extension of $L_2(D)$ inner product. Let f be any element of K^\perp . Extend f by zero in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$.

Let $w(x, t)$ be the solution of

$$(5.15) \quad \begin{aligned} L^*w &= f \quad \text{in } (\mathbf{R}^n \times (0, T)) \setminus \Omega, \\ w|_{t=T} &= 0, \quad w|_{\partial\Omega} = 0, \end{aligned}$$

Note that $w(x, t) \in C((0, T), \mathring{H}_1(\mathbf{R}^n \setminus \Omega(t)))$.

By the Green formula we have

$$0 = (v, f) = (v, L^*w) = ih \int_{\mathbf{R}^n \setminus \Omega(0)} v(x, 0) \overline{w}(x, 0) dx,$$

where $v(x, t)$ is a physically meaningful solution. Since $v(x, 0) = 0$ in $\Omega_0 \setminus \Omega(0)$ and $v(x, 0)$ is arbitrary in $\mathbf{R}^n \setminus \Omega_0$, we get that

$$w(x, 0) = 0, \quad x \in \mathbf{R}^n \setminus \Omega_0.$$

Consider $w(x, t)$ in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$. We assume that the electric potential $V(x, t) = 0$ in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$. If $n = 3$ or $n = 2$ and the total magnetic

flux $\frac{e}{hc} \int_{\partial\Omega_0} A(x, t) \cdot dx = 0$, we can choose the gauge such that $A(x, t) = 0$ in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$. In this case the equation (5.1) has the form

$$(5.16) \quad ih \frac{\partial w}{\partial t} + \frac{h^2}{2m} \Delta w = 0,$$

for $(x, t) \in (\mathbf{R}^n \setminus \Omega_0) \times (0, T)$.

When $n = 2$ and the total magnetic flux is not zero we can choose the gauge to make $A(x, t)$ equal to AB potential in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$ (cf. [AB]). Then in polar coordinates (r, θ) we have in $(R^2 \setminus \Omega_0) \times (0, T)$:

$$(5.17) \quad ih \frac{\partial w}{\partial t} + \frac{h^2}{2m} \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} + i\alpha \right)^2 w(r, \theta, t) \right] = 0.$$

Lemma 5.1. *Let $w(x, t)$ be the solution of (5.1) in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$, $w(x, 0) = w(x, T) = 0$, $x \in \mathbf{R}^n \setminus \Omega_0$, and $w(x, t) \in C((0, T), L_2(\mathbf{R}^n \setminus \Omega_0))$. Then $w = 0$ in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$.*

Proof: Consider the case of equation (5.17). The case of the equation (5.16) for $n = 2$ or $n = 3$ is similar. Let R be such that $\overline{\Omega_0} \subset B_R = \{x : |x| < R\}$. Extend $w(x, t)$ by zero for $t > T$ and $t < 0$. Making the Fourier transform in t we get for $|x| > R$

$$(5.18) \quad -h\xi_0 \tilde{w}(r, \theta, \xi_0) + \frac{h^2}{2m} \left[\frac{\partial^2 \tilde{w}(r, \theta, \xi_0)}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{w}}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} + i\alpha \right)^2 \tilde{w} \right] = 0,$$

where $\theta \in [0, 2\pi]$, $r = |x| > R$. Since $\int_{\mathbf{R}^2 \setminus \Omega_0} \int_0^T |w(x, t)|^2 dx dt < +\infty$ we have that $\tilde{w}(x, \xi_0)$ is continuous in ξ_0 and $\int_{|x| > R} |\tilde{w}(x, \xi_0)|^2 dx < +\infty$ for any $\xi_0 \in \mathbf{R}$.

The general solution of (5.18) in $|x| > R$ has the form (see [AB])

$$(5.19) \quad \tilde{w}(x, \xi_0) = \sum_{n=-\infty}^{\infty} w_n(r, \xi_0) e^{in\theta},$$

where

$$w_n(r, \xi_0) = a_n(\xi_0) J_{m+\alpha}(kr) + b_n(\xi_0) J_{-m-\alpha}(kr), \quad k = \sqrt{\frac{2m}{h}(-\xi_0)}.$$

We have

$$\int_{|x| > R} |\tilde{w}(x, \lambda)|^2 dx = \sum_{n=-\infty}^{\infty} \int_{r > R} |w_n(r, \xi_0)|^2 r dr.$$

It follows from (5.19) that $\int_{r>R} |w_n(r, \xi_0)|^2 dr < +\infty$ for all ξ_0 iff $a_n(\xi_0) = b_n(\xi_0) = 0$. Therefore $w(x, t) = 0$ for $|x| > R$, $t \in [0, T]$. Using the unique continuation property (cf. [I]) we get that $w(x, t) = 0$ in $(\mathbf{R}^2 \setminus \Omega_0) \times (0, T)$. \square

Lemma 5.2. *Any $u(x, t) \in W(D)$ can be approximated in the V^* norm by physically meaningful solutions $v(x, t)$, i.e. by $v(x, t) \in C((0, T), H_2(\mathbf{R}^n \setminus \Omega(t)) \cap \mathring{H}_1(\mathbf{R}^n \setminus \Omega(t)))$ that satisfy (5.1) in $(\mathbf{R}^n \times (0, T)) \setminus \Omega$ with the boundary conditions (5.13) and the initial conditions (5.14) where $u_0(x) = 0$ in $\Omega_0 \setminus \Omega(0)$.*

Proof: Let $u(x, t) \in W(D)$. We have

$$(u, f)_D = (u, L^*w)_D,$$

where $(u, f)_D$ is the inner product in $L_2(D)$. Note that $u|_{\partial\Omega} = 0$ and $w|_{\partial\Omega} = 0$. Also $u|_{t=0} = 0$ and $w|_{t=T} = 0$ in D . By Lemma 5.1 $w = 0$ in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$. Therefore $w|_{\partial\Omega_0 \times [0, T]} = 0$ and the restriction of the normal derivative of w to $\partial\Omega_0 \times [0, T]$ is equal to zero in the distribution sense (cf. [E6], §24). Hence applying the Green formula over D we get $(Lu, w)_D = (u, L^*w)_D = 0$ since $Lu = 0$ and all boundary terms are equal to zero. Therefore $(u, f) = (u, L^*w) = 0$ for any $f \in K^\perp$. Thus $u \in K$, i.e. u can be approximated in the norm of V^* by the physically meaningful solutions $v_\varepsilon : \|u - v_\varepsilon\|_{V^*} = \max_{t \leq T} \int_{D_t} |u - v_\varepsilon|^2 dx < \varepsilon$ where $\varepsilon > 0$ can be chosen arbitrary small. \square

If, for example, $|u(x, t)|^2 = 4 \sin^2 \frac{\alpha}{2} + O(\varepsilon_1)$ in a small neighborhood U_0 of a point P_0 , then $\int_{U_0} |u(x, t)|^2 dx = 4 \sin^2 \frac{\alpha}{2} \mu(U_0) + O(\varepsilon) \mu(U_0)$, where $\mu(U_0)$ is the volume of the neighborhood U_0 . Choose ε much smaller than ε_1 . We get by Lemma 5.2 that

$$(5.20) \quad \int_{U_0} |v_\varepsilon(x, t)|^2 dx = 4 \sin^2 \frac{\alpha}{2} \mu(U_0) + O(\varepsilon_1) \mu(U_0),$$

i.e. we can determine the flux α by the measurement of a physically meaningful solution.

5.5 A new inverse problem for the time-dependent Schrödinger equations

Let $(\Omega_0 \times [0, T]) \setminus \Omega$ be the same domains as before. Let

$$ih \frac{\partial u_p}{\partial t} - \frac{1}{2m} \sum_{j=1}^n \left(-ih \frac{\partial u_p}{\partial x_j} - \frac{e}{c} A_j^{(p)}(x, t) \right)^2 u_p(x, t) - eV^{(p)}(x, t)u_p(x, t) = 0$$

be two Schrödinger equations in $(\mathbf{R}^n \times (0, T)) \setminus \Omega$ with electromagnetic potentials $A^{(p)}(x, t), V^{(p)}(x, t)$, $p = 1, 2$. Suppose that supports of $A^{(p)}(x, t)$ and $V^{(p)}(x, t)$ are contained in $(\Omega_0 \times (0, T)) \setminus \Omega$, $p = 1, 2$.

Theorem 5.3. *Suppose*

$$(5.21) \quad \begin{aligned} u_p(x, 0) &= f_0(x), \quad x \in \mathbf{R}^n \setminus \Omega(0), \quad p = 1, 2, \quad f_0(x) = 0 \text{ if } x \in \Omega_0 \setminus \Omega(0), \\ u_1(x, T) &= u_2(x, T), \quad x \in \mathbf{R}^n \setminus \Omega_0, \end{aligned}$$

for all $f_0(x) \in L_2(\mathbf{R}^n \setminus \Omega_0)$. Suppose that geometric conditions on the obstacles $\Omega(t)$ of Theorems 1.1 and 1.2 in [E1] hold. Then there exists a gauge $g(x, t) \in C^\infty((\mathbf{R}^n \times (0, T)) \setminus \Omega)$, $|g(x, t)| = 1$ in $(\mathbf{R}^n \times (0, T)) \setminus \Omega$, $g(x, t) = 1$ for $(x, t) \in (\mathbf{R}^n \setminus \Omega_0) \times (0, T)$ such that $(A^{(1)}(x, t), V^{(1)}(x, t))$ and $(A^{(2)}(x, t), V^{(2)}(x, t))$ are gauge equivalent.

Proof: Denote $v(x, t) = u_1(x, t) - u_2(x, t)$. Then $v(x, 0) = v(x, T) = 0$ for $x \in \mathbf{R}^n \setminus \Omega_0$ and $v(x, t)$ satisfies

$$ih \frac{\partial v}{\partial t} + \frac{h^2}{2m} \Delta v(x, t) = 0 \text{ in } (\mathbf{R}^n \setminus \Omega_0) \times (0, T).$$

By Lemma 5.1 $v(x, t) = 0$ in $(\mathbf{R}^n \setminus \Omega_0) \times (0, T)$. Therefore $v|_{\partial\Omega_0 \times (0, T)} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial\Omega_0 \times (0, T)} = 0$, where $\frac{\partial}{\partial \nu}$ is the normal derivative. Thus $u_1|_{\partial\Omega_0 \times (0, T)} = u_2|_{\partial\Omega_0 \times (0, T)}$ and $\frac{\partial u_1}{\partial \nu}|_{\partial\Omega_0 \times (0, T)} = \frac{\partial u_2}{\partial \nu}|_{\partial\Omega_0 \times (0, T)}$. By Lemma 5.2 the restrictions of u_p to $\partial\Omega_0 \times (0, T)$ are dense in $H_{-\frac{1}{2}}(\partial\Omega_0 \times (0, T))$.

Note that these restrictions exist by the partial hypoellipticity property (cf., for example, [E6], §24). Therefore the Dirichlet-to-Neumann operators Λ_1 and Λ_2 are equal on $\partial\Omega_0 \times (0, T)$. Also $u_1(x, 0) = u_2(x, 0) = 0$ on $\Omega_0 \setminus \Omega(0)$. Then it follows from [E1] that $(A^{(1)}(x, t), V^{(1)}(x, t))$ and $(A^{(2)}(x, t), V^{(2)}(x, t))$ are gauge equivalent.

Remark 5.2 Suppose $n = 2$ and

$$\alpha_p = \frac{e}{hc} \int_{\partial\Omega_0} A^{(0)}(x, t) \cdot dx$$

are not zero, $p = 1, 2$. We assume, in addition to (5.21), that $\alpha_1 = \alpha_2 = \alpha$ is a constant and the equation (5.1) has the form (5.17) for $(\mathbf{R}^2 \setminus \Omega_0) \times (0, T)$. Then Theorem 5.3 holds since we can apply Lemma 5.1 to the equation (5.17) in $(\mathbf{R}^2 \setminus \Omega_0) \times (0, T)$.

6 The gravitational AB effect

6.1 Global isometry

We shall start with a short summary of the magnetic AB effect: consider the Schrödinger equation (1.1) in $\mathbf{R}^2 \setminus \Omega_1$ with the boundary condition (1.2). We assume that the magnetic field $B = \text{curl } A$ is zero in $\mathbf{R}^2 \setminus \Omega_1$. Then locally in any simply-connected neighborhood $U \subset \mathbf{R}^2 \setminus \Omega_1$ the magnetic potential is gauge equivalent to a zero potential and $\int_\gamma A \cdot dx = 0$ for any closed curve $\gamma \subset U$. However, globally in $\mathbf{R}^2 \setminus \Omega_1$ the magnetic potential $A(x)$ may be not gauge equivalent to a zero potential, in particular, $\int_\gamma A \cdot dx = \alpha$ may be not zero if γ is a closed curve in $\mathbf{R}^2 \setminus \Omega_1$ encircling Ω_1 .

The fact that the magnetic potential $A(x)$ is not gauge equivalent globally in $\mathbf{R}^2 \setminus \Omega_1$ to the zero potential has a physical impact, and this phenomenon is called the Aharonov-Bohm effect. More generally, if A_1 and A_2 are not gauge equivalent then each of them makes a distinct physical impact. Similar situation (local versus global) appears in different branches of mathematical physics.

Consider, for example, a pseudo-Riemannian metric $\sum_{j,k=0}^n g_{jk}(x) dx_j dx_k$ with Lorentz signature in Ω , where $x_0 \in \mathbf{R}$ is the time variable, $x = (x_1, \dots, x_n) \in \Omega$, $\Omega = \Omega_0 \setminus \cup_{j=1}^m \overline{\Omega}_j$, Ω_0 is simply connected, $\overline{\Omega}_j \subset \Omega_0$, Ω_j , $1 \leq j \leq m$, are obstacles (cf. subsection 3.3). We assume that $g_{jk}(x)$ are independent of x_0 , i.e. the metric is stationary.

Consider a group of transformations (changes of variables)

$$(6.1) \quad \begin{aligned} x' &= \varphi(x), \\ x'_0 &= x_0 + a(x), \end{aligned}$$

where $x' = \varphi(x)$ is a diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega'} = \varphi(\overline{\Omega})$ and $a(x) \in C^\infty(\overline{\Omega})$. Two metrics $\sum_{j,k=0}^n g_{jk}(x)dx_jdx_k$ and $\sum_{j,k=0}^n g'_{jk}(x)dx'_jdx'_k$ are called isometric if

$$(6.2) \quad \sum_{j,k=0}^n g_{jk}(x)dx_jdx_k = \sum_{j,k=0}^n g'_{jk}(x)dx'_jdx'_k,$$

where (x'_0, x') and (x_0, x) are related by (6.1).

The group of isomorphisms will play the same role as the gauge group for the magnetic AB effect.

Let

$$\square_g u(x_0, x) = 0 \quad \text{in } \mathbf{R} \times \Omega$$

be the wave equation corresponding to the metric g , i.e.

$$(6.3) \quad \square_g u \stackrel{def}{=} \sum_{j,k=0}^n \frac{1}{\sqrt{(-1)^n g_0}} \frac{\partial}{\partial x_j} \left(\sqrt{(-1)^n g_0} g^{jk}(x) \frac{\partial u}{\partial x_k} \right) = 0,$$

where $g_0 = \det[g_{jk}]_{j,k=0}^n$, $[g^{jk}(x)] = [g_{jk}]^{-1}$.

Solutions of (6.3) are called gravitational waves on the background of the space-time with the metric g .

Consider the initial boundary value problem for (6.3) in $\mathbf{R} \times \Omega$ with zero initial conditions

$$(6.4) \quad u(x_0, x) = 0 \quad \text{for } x_0 \ll 0, \quad x \in \Omega,$$

and the boundary condition

$$(6.5) \quad u|_{\mathbf{R} \times \partial\Omega_0} = f, \quad u|_{\mathbf{R} \times \partial\Omega_j} = 0, \quad 1 \leq j \leq m,$$

where $f \in C_0^\infty(\mathbf{R} \times \partial\Omega_0)$. Let Λ_g be the Dirichlet-to-Neumann (DN) operator, i.e. $\Lambda_g f = \frac{\partial u}{\partial \nu_g}|_{\mathbf{R} \times \partial\Omega_0}$, where

$$(6.6) \quad \frac{\partial u}{\partial \nu_g} = \sum_{j,k=0}^n g^{jk}(x) \nu_j(x) \frac{\partial u}{\partial x_k} \left(\sum_{p,r=0}^n g^{pr}(x) \nu_p \nu_r \right)^{-\frac{1}{2}}.$$

Here $u(x_0, x)$ is the solution of (6.3), (6.4), (6.5), $\nu(x) = (\nu_1, \dots, \nu_n)$ is the outward unit normal to $\partial\Omega_0$, $\nu_0 = 0$.

Let Γ be an open subset of $\partial\Omega_0$. We shall say that boundary measurements are taken on $(0, T) \times \Gamma$ if we know the restriction $\Lambda_g f|_{(0, T) \times \Gamma}$ for any $f \in C_0^\infty((0, T) \times \Gamma)$.

Consider metric g' in Ω' and the corresponding initial-boundary value problem

$$(6.7) \quad \square_{g'} u'(x'_0, x') = 0 \quad \text{in } \mathbf{R} \times \Omega',$$

$$(6.8) \quad u'(x'_0, x') = 0 \quad \text{for } x'_0 \ll 0, \quad x' \in \Omega',$$

$$(6.9) \quad u|_{\mathbf{R} \times \partial\Omega'_0} = f, \quad u'|_{\mathbf{R} \times \partial\Omega'_j} = 0, \quad 1 \leq j \leq m',$$

where $\Omega' = \Omega'_0 \setminus \bigcup_{j=1}^m \overline{\Omega'_j}$.

We assume that $\partial\Omega_0 \cap \partial\Omega'_0 \neq \emptyset$. Let Γ be an open subset of $\partial\Omega_0 \cap \partial\Omega'_0$.

The following theorem was proven in [E2] (see [E2], Theorem 2.3).

Theorem 6.1. *Suppose $g^{00}(x) > 0$, $g_{00}(x) > 0$ in $\overline{\Omega}$ and $(g')^{00} > 0$, $g'_{00} > 0$ in $\overline{\Omega'}$. Suppose $\Lambda_g f|_{(0, T) \times \Gamma} = \Lambda_{g'} f|_{(0, T) \times \Gamma}$ for all $f \in C_0^\infty((0, T) \times \Gamma)$. Suppose $T > T_0$, where T_0 is sufficiently large. Then metrics g and g' are isometric, i.e. there exists a change of variables (6.1) such that (6.2) holds. Moreover, $\varphi|_\Gamma = I$, $a|_\Gamma = 0$.*

If two metrics g and g' in Ω and Ω' , respectively, are isometric, then the solutions $u(x_0, x)$ and $u(x'_0, x')$ of the corresponding wave equations are the same after the change of variables (6.1). Therefore isometric metrics have the same physical impact.

Let g and g' be two stationary metrics in Ω and Ω' , respectively. Let V be a neighborhood such that $\overline{V} \cap \partial\Omega_0 \supset \Gamma \neq \emptyset$. Suppose g and g' are isometric in V , i.e. there exists a change of variables

$$\begin{aligned} x' &= \varphi_V(x), \quad x \in \overline{V}, \\ x'_0 &= x_0 + a_V(x), \quad x \in \overline{V}, \end{aligned}$$

such that (6.2) holds for $x \in \overline{V}$. We want to find out what is the impact of g and g' being not isometric. One can find a change of variables of the form (6.1) to replace g' in Ω' by an isometric metric \hat{g} in $\hat{\Omega}$ such that $\Omega_0 \cap \hat{\Omega}_0 \supset V$ and $g = \hat{g}$ in \overline{V} .

It follows from Theorem 6.1 that g and \hat{g} are not isometric if and only if the boundary measurements

$$\Lambda_g f|_{(0, T) \times \Gamma} \neq \Lambda_{\hat{g}} f|_{(0, T) \times \Gamma} \quad \text{for some } f \in C_0^\infty((0, T) \times \Gamma),$$

i.e. metrics g and \hat{g} (and therefore g and g') have different physical impact. This fact (i.e. that non-isometric metric have a different physical impact) is called the gravitational AB effect.

Note that the open set Γ can be arbitrary small. However the time interval $(0, T)$ must be large enough: $T > T_0$.

6.2 Locally static stationary metrics

Let g and g' be isometric. Substituting $dx'_0 = dx_0 + \sum_{j=1}^n a_{x_j}(x)dx_j$ and taking into account that dx_0 is arbitrary, we get from (6.1) and (6.2) that

$$(6.10) \quad g'_{00}(x') = g_{00}(x),$$

$$(6.11) \quad 2g'_{00}(x') \sum_{j=1}^n a_{x_j}(x)dx_j + 2 \sum_{j=1}^n g'_{j0}(x')dx'_j = 2 \sum_{j=1}^n g_{j0}(x)dx_j.$$

Using (6.10) we can rewrite (6.11) in the form

$$(6.12) \quad \sum_{j=1}^n \frac{1}{g'_{00}(x')} g'_{j0}(x')dx'_j = \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x)dx_j - \sum_{j=1}^n a_{x_j}(x)dx_j.$$

Let γ be an arbitrary closed curve in Ω , and let γ' be the image of γ in Ω' under the map (6.1). Integrating (6.12) we get

$$(6.13) \quad \int_{\gamma'} \sum_{j=1}^n \frac{1}{g'_{00}(x')} g'_{j0}(x')dx'_j = \int_{\gamma} \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x)dx_j,$$

since $\int_{\gamma} \sum_{j=1}^n a_{x_j}(x)dx_j = 0$. Therefore the integral

$$(6.14) \quad \alpha = \int_{\gamma} \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x)dx_j$$

is the same for all isometric metrics. □

A stationary metric g is called static in Ω if it has the form

$$(6.15) \quad g_{00}(x)(dx_0)^2 + \sum_{j,k=1}^n g_{jk}(x)dx_j dx_k,$$

i.e. when $g_{0j}(x) = g_{j0}(x) = 0$, $1 \leq j \leq n$, $x \in \Omega$.

We assume that g is stationary in Ω and locally static, i.e. for any simply-connected neighborhood $U \subset \Omega$ there exists a change of variables $x' = \varphi_U(x)$, $x'_0 = x_0 + a_U(x)$, $x \in \overline{U}$ that transform $g|_U$ to a static metric. Therefore (6.13) implies that $\int_\gamma \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x) dx_j = 0$ for any closed curve $\gamma \subset U$. Suppose the metric g is not static globally in Ω . Then integral (6.14) may be not zero. It plays a role of magnetic flux for the magnetic AB effect and α in (6.14) depends only on the homotopy class of γ .

Let $V \subset \Omega$, $\overline{V} \cap \partial\Omega_0 \supset \Gamma \neq \emptyset$ and let g'_V be a static metric in V isometric to g for $x \in V$. Denote by g' an arbitrary extension of g'_V to $\overline{\Omega}$ such that g' is static and stationary. We have that g and g' are isometric in V , and we assume that g and g' are not isometric in $\overline{\Omega}$. Then the Theorem 6.1 implies that the boundary measurements for g and g' on $\Gamma \times (0, T)$ are not equal for the same $f \in C_0^\infty(\Gamma \times (0, T))$. Therefore g and g' have different physical impact, i.e. the gravitational AB effect holds. \square

The gravitational AB effect for a special class of locally static metrics was considered previously in [E].

6.3 A new inverse problem for the wave equation

Let g and g' be two stationary metrics in $\mathbf{R}^n \setminus \cup_{j=1}^m \Omega_j$ such that

$$(6.16) \quad g_{jk}(x) = g'_{jk}(x) \quad \text{for } |x| > R,$$

where R is large. Assume also that

$$(6.17) \quad g_{jk}(x) = \eta_{jk} + h_{jk}(x) \quad \text{for } |x| > R,$$

where

$$\sum_{j,k=1}^n \eta_{jk} dx_j dx_k = dx_0^2 - \sum_{j=1}^n dx_j^2$$

is the Minkowski metric and $h_{jk}(x) = O(\frac{1}{|x|^{1+\varepsilon}})$, $\varepsilon > 0$, for $|x| > R$.

The following theorem is analogous to Theorem 5.3.

Theorem 6.2. *Let $\square_g u = 0$ and $\square_{g'} u' = 0$ in $(0, T) \times (\mathbf{R}^n \setminus \cup_{j=1}^m \Omega_j)$, where $T > T_0$ (cf. Theorem 6.1). Consider two initial-boundary value prob-*

lems

$$\begin{aligned} u(0, x) &= u_0(x), & u'(0, x) &= u_0(x), \\ u_t(0, x) &= u_1(x), & u'_t(0, x) &= u_1(x), & x \in \mathbf{R}^n \setminus \bigcup_{j=1}^n \overline{\Omega_j}, \\ u|_{\partial\Omega_j \times (0, T)} &= 0, & u'|_{\partial\Omega_j \times (0, T)} &= 0, & 1 \leq j \leq m, \\ u_0(x) &= u_1(x) = 0 & \text{in } B_R \setminus \bigcup_{j=1}^m \Omega_j, \end{aligned}$$

where $B_R = \{x : |x| < R\}$. Suppose $g_{00}(x) > 0$, $g'_{00}(x) > 0$, $g^{00}(x) > 0$, $(g')^{00} > 0$ in $\mathbf{R}^n \setminus \bigcup_{j=1}^n \Omega_j$. If $u_0(x) \in \dot{H}_1(\mathbf{R}^n \setminus B_R)$, $u_1(x) \in L_2(\mathbf{R}^n \setminus B_R)$ and if

$$u(T, x) = u'(T, x), \quad u_{x_0}(T, x) = u'_{x_0}(T, x), \quad x \in \mathbf{R}^n \setminus B_R,$$

for all $u_0(x)$ and $u_1(x)$, then metrics g and g' are isometric in $\mathbf{R}^n \setminus \bigcup_{j=1}^n \Omega_j$.

Proof: It follows from the existence and uniqueness theorem that the solutions $u(x_0, x)$ and $u'(x_0, x)$ belong to $H_1((0, T) \times (\mathbf{R}^n \setminus \bigcup_{j=1}^n \Omega_j))$. Let $v = u(x_0, x) - u'(x_0, x)$. Then $\square_g v = 0$ in $(0, T) \times (\mathbf{R}^n \setminus B_R)$ and $v(0, x) = v_{x_0}(0, x) = 0$, $v(T, x) = v_{x_0}(T, x) = 0$ for $x \in \mathbf{R}^n \setminus B_R$. Extend $v(x_0, x)$ by zero for $x_0 > T$ and $x_0 < 0$ and make the Fourier transform in x_0 : $\tilde{v}(\xi_0, x) = \int_{-\infty}^{\infty} v(x_0, x) e^{-ix_0 \xi_0} dx_0$. Then $\tilde{v}(\xi_0, x)$ belongs to $L_2(\mathbf{R}^n \setminus B_R)$ for all $\xi_0 \in \mathbf{R}$ and satisfies the equation

$$L(i\xi_0, \frac{\partial}{\partial x}) \tilde{v}(\xi_0, x) = 0, \quad x \in \mathbf{R}^n \setminus B_R,$$

where $L(i\xi_0, i\xi)$ is the symbol of \square_g .

It follows from [H] that $\tilde{v}(\xi_0, x) = 0$ in $\mathbf{R}^n \setminus B_R$ for all ξ_0 . Therefore $u(x_0, x) = u'(x_0, x)$ for $x_0 \in (0, T)$, $x \in \mathbf{R}^n \setminus B_R$. Then $u|_{(0, T) \times \partial B_R} = u'|_{(0, T) \times \partial B_R}$ and $\frac{\partial u}{\partial \nu_g}|_{(0, T) \times \partial B_R} = \frac{\partial u'}{\partial \nu_g}|_{(0, T) \times \partial B_R}$, i.e. the boundary measurements of u and u' on $(0, T) \times \partial B_R$ are the same.

Analogously to the proof of Lemma 5.2 one can show that $u|_{(0, T) \times \partial B_R}$ and $u'|_{(0, T) \times \partial B_R}$ are dense in $H_{-\frac{1}{2}}((0, T) \times \partial B_R)$. Hence the DN operators Λ and Λ' are equal on $(0, T) \times \partial B_R$. Thus Theorem 6.1 implies that g and g' are isometric.

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